# Nonlinear Viscoelastic Analysis of Laminated Composite Plates – A Multi Scale Approach

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**ABSTRACT**

*Laminated composite plates are widely used in modern structures. Resins of composites are almost made of polymers which show time dependent and and in some cases stress dependent behaviour. In this paper, a laminated composite plate is analysed using a multiscale method. At first, material properties of a lamina is obtained using an analytical micromechanical approach called simplified unit cell method (SUCM) and then in macromechanical level, Generalized Differential Quadrature Method (GDQM) is used to analyse laminated composite plate. Schapery's integral is used to model nonlinear viscoelastic behaviour of the matrix. Prony series is considered to define the compliance of matrix. Micromechanical process includes obtaining overall properties of the composite by SUCM. Both geometrical and material nonlinearity are taken into account in order to multiscale analysis of laminated composite plate.*

**KEYWORDS**

*nonlinear viscoelastic, Multi-scale ana;ysis, nonlinear analysis, laminated composites, generalized differential quadrature method*

## 1. INTRODUCTION

Fiber reinforced polymers (FRPs) are widely used in engineering structures. Time dependency is inherent in polymer material behaviour. In addition to the, time dependency, some of them also show stress dependency behaviour, which is called nonlinear visoelastic behaviour. Nonlinear viscoelastic behaviour of materials can be modeled using Schapery single integral model, which is based on irreversible thermodynamics [1]. Haj-Ali and Muliana [2-3] used Schapery single integral to model nonlinear viscoelastic behaviour. They used a recursive–iterative method for the numerical integration of Schapery model. Method of cells (MOC) was first introduced by Aboudi [4] and used for different applications such as thermo-elastic and linear viscoelastic analysis of composites [5]. Tuttle and Brinson [6] conducted creep-recovery test for off-axis graphite (T300)/epoxy (5208) composites; the nonlinear viscoelastic behaviour is considered to analysis of graphite/epoxy laminates subjected to the in-plane loading, using classical laminated plate theory (CLPT). With same basic assumptions as MOC, a simplified unit cell method (SUCM) was presented by Aghdam et al [7] to study the collapse behaviour of metal matrix composites in combined loading conditions. SUCM is used by Masoumi et al [8] for multi-scale analysis of laminated composite plates. Shu and Richrds[9] introduced a recursive method to find weighted

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coefficients named generalized differential quadrature method (GDQM). The basic idea of the GDQM is that any derivative at a mesh point can be approximated by a linear summation of all the functional values on nodes along a mesh line. In the present work, both geometrical and material nonlinearity is considered to analysis of laminated composite plates. MOC [3] and experimental results [6] are used to validate the micro model; while FEM is used to validate elastic structural analysis. In the present work, nonlinear viscoelastic behaviour of laminated composite plates is considered using a multi-scale method including GDQM in macro and SUCM in micro level. Considering simplicity, accuracy and compatibility with multi scale analysis SUCM [7, 8] and GDQM are chosen as an appropriate method for this work. Effects of different parameters in nonlinear viscoelastic behaviour of plate are discussed.

## 2. NONLINEAR VISCOELASTIC MATERIAL

Nonlinear viscoelastic behaviour can be expressed using Schapery model as [1]:

$$\varepsilon(t) = g_0^{\sigma(t)} D_0 \sigma(t) + g_1^{\sigma(t)} \int_0^t \Delta D(\psi^t - \psi^\tau) \frac{\partial (g_2^{\sigma(\tau)} \sigma(\tau))}{\partial \tau} d\tau \tag{1}$$

Where $\psi^t$ and $\psi^\tau$ are reduced times:

$$\psi^t \equiv \psi(t) = \int_0^t \frac{d\xi}{a^{\sigma(\xi)}} \tag{2}$$

$D_0$ is instantaneous compliance at the beginning of analysis and $\Delta D(t)$ is transient compliance which can be defined using Prony series:

$$\Delta D(\psi^t) = \sum_{n=1}^{N} D_n (1 - \exp(-\lambda_n \psi^t)) \tag{3}$$

in which $D_n$ are coefficients of Prony series and are determined by fitting linear viscoelastic behaviour of materials; $\lambda_n$ is reciprocal of the retardation time and nonlinear stress dependent parameters can be defined as follow[2]:

$$g_i = 1 + \sum_{n=1}^{N_{g_i}} \alpha_{ni} \left\langle \frac{\sigma_{eq}}{\sigma_0} - 1 \right\rangle^n \quad , \quad a_\sigma = 1 + \sum_{n=1}^{Na_\sigma} \beta_n \left\langle \frac{\sigma_{eq}}{\sigma_0} - 1 \right\rangle^n$$

$$\langle X \rangle = \begin{cases} 0 & X \le 0 \\ 1 & X > 0 \end{cases} \quad , \quad \sigma_{eq}^2 = \frac{3}{2} S_{ij} S_{ij} \tag{4}$$

where $(\alpha_{ni}, \beta_n)$ are calibrated coefficients and $\sigma_0$ is the limit stress which higher than that material shows nonlinear viscoelastic behaviour.

## 3. MICROMECHANICAL ANALYSIS

It is assumed that the effective properties of the composite system are presented of those of the RVE. As a result of this periodic arrangement, it is logical to analysis a quarter of the RVE. This cell is divided into four sub-cells (Fig. 1). The first sub-cell is a fiber constituent, while sub-cells 2, 3, and 4 represent the matrix constituents.

It is shown in Fig. 1 that the continuous square fibers extended in the $X_1$ direction. The total area of the cell and the width are considered unit. It is assumed that the interfaces between adjacent sub-cells are perfectly bonded. In the SUCM the displacement components are linear which results in constant stress and strain within the sub-cells. Besides, for in simplified unit cell method, it is considered that no shear stress is induced inside the sub-cells from the normal stress on the RVE in material coordinates [7]. Equilibrium equations in each direction are [8]:

$$\begin{aligned} & a^2\sigma_{11}^{(1)} + ab\,\sigma_{11}^{(2)} + ab\,\sigma_{11}^{(3)} + b^2\sigma_{11}^{(4)} = \overline{\sigma}_{11}(a+b)^2 \\ & a\sigma_{22}^{(2)} + b\,\sigma_{22}^{(4)} = \overline{\sigma}_{22}(a+b) \\ & a\sigma_{33}^{(3)} + b\,\sigma_{33}^{(4)} = \overline{\sigma}_{33}(a+b) \end{aligned} \tag{5}$$

where the superscripts denote the sub-cells number. Equilibrium in the interfaces leads to:

$$\begin{aligned} & \sigma_{22}^{(1)} = \sigma_{22}^{(2)} \qquad \sigma_{22}^{(3)} = \sigma_{22}^{(4)} \\ & \sigma_{33}^{(1)} = \sigma_{33}^{(3)} \qquad \sigma_{33}^{(2)} = \sigma_{33}^{(4)} \end{aligned} \tag{6}$$

Compatibility of the displacements in the RVE can be written in the following form:

$$\begin{aligned} & \varepsilon_{11}^{(1)} = \varepsilon_{11}^{(2)} = \varepsilon_{11}^{(3)} = \varepsilon_{11}^{(4)} = \overline{\varepsilon}_{11} \\ & a\varepsilon_{22}^{(1)} + b\varepsilon_{22}^{(2)} = a\varepsilon_{22}^{(3)} + b\varepsilon_{22}^{(4)} = \overline{\varepsilon}_{22}(a+b) \\ & a\varepsilon_{33}^{(1)} + b\varepsilon_{33}^{(2)} = a\varepsilon_{33}^{(3)} + b\varepsilon_{33}^{(4)} = \overline{\varepsilon}_{33}(a+b) \end{aligned} \tag{7}$$

Constitutive equations for the isotropic viscoelastic matrix sub-cells:

$$\begin{aligned} & \varepsilon_{ij}^{k}(t) = (1+\nu_m)S_c^{k}(t)\sigma_{ij}^{k}(t)\delta_{\alpha\beta} - \nu_m S_c^{k}(t)\sigma_{ij}^{k}(t) + \\ & (1+\nu_m)R_{ij}^{k}(t) - \nu_m R_{ij}^{k}(t) \qquad k = 2,3,4 \end{aligned} \tag{8}$$

Using Eqs. 5 to 8, stresses in sub-cells can be found by solving following equation:

$$A\sigma = F + H \tag{9}$$

Where $A$ is material matrix, $\sigma$ is unknown sub-cells stress vector, $F$ is external load vector, and H is hereditary vector, which are defined in detailed in reference [8]. Once sub-cells stresses become known, average strain and then material properties are also known[8].

## 4. STRUCTURAL ANALYSES

In structural scale, first order shear deformation theory (FSDT) is applied to derive laminated plate equations of motion. Displacement field based on FSDT can be expressed as:

$$\begin{aligned} U(x,y,z) &= u(x,y) + z\varphi_x(x,y) \\ V(x,y,z) &= v(x,y) + z\varphi_y(x,y) \\ W(x,y,z) &= w(x,y) \end{aligned} \tag{10}$$

where $u$, $v$ and $w$ denote displacements in $x$, $y$ and $z$ directions, respectively. $\varphi_x$ and $-\varphi_y$ are angular displacements about $x$ and $y$ axis, respectively. Taking into account Von-Karman nonlinearity type to express strains definition and using them into the constitutive equations of laminated composite plate and finally replacing resultant stresses into the equations of equilibrium leads to the equations of motion of laminated composite plate [10].

### 4.1 Applying GDQM

GDQM approximates the derivative of a function at any point by linear summation of all functional values along a mesh line. Derivatives of a function like $w(x, y)$ can be calculated as

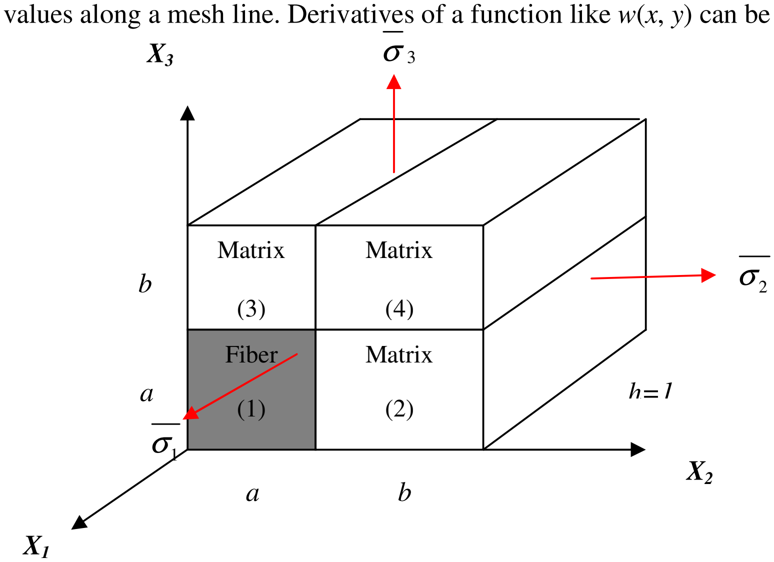


Fig. 1 RVE for analytical model.

follow [9],

$$w_{,x}^{(m)}(x_i, y_j) = \sum_{k=1}^{N_x} a_{ik}^{(m)} f(x_k, y_j) \qquad i = 1, 2, ..., N_x \tag{11}$$

$$w_{,y}^{(n)}(x_i, y_j) = \sum_{l=1}^{N_y} b_{il}^{(n)} f(x_i, y_l) \qquad j = 1, 2, ..., N_y \tag{12}$$

$$w_{,xy}^{(m+n)}(x_i, y_j) = \sum_{l=1}^{N_y} \sum_{k=1}^{N_x} a_{ik}^{(m)} b_{jl}^{(n)} f(x_k, y_l) \tag{13}$$

based on GDQM, Weighted coefficients are determined in the process outlined in [9]: where $l_x$ and $l_y$ are lengths along x and y directions, respectively. Once derivatives and boundary conditions [10] are applied to the displacement based equations of motions, the results would be a system of $5(n_x \times n_y)$ nonlinear algebraic. Newton–Raphson method is used to solve the resulted non-linear system of equations. For example, the first equation of motions after applying GDQM is shown in Appendix A.

Where constants are defined as follow [10]:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{t}{2}}^{\frac{t}{2}} Q_{ij}(1, z, z^2) dz \qquad i, j = 1, 2, 6 \tag{15}$$

## 5. MULTI SCALE ANALYSES

Multi-scale analysis is applied as follow. At first, stresses are calculated in elastic laminated plate; then nonlinear parameters are calculated using these stresses. Once nonlinear parameters are determined, the iterative method for nonlinear analysis of nonlinear viscoelastic composite plates begins. In micromechanical level, material properties of each node in each lamina are calculated using different RVEs. In each RVE, if the new sub-cells stresses converge, then the material properties are saved, otherwise the micromechanical approach would be repeated with new parameters until the stresses converge; then the new material properties would be used in five nonlinear equations of motion of laminated composite plate; next, nonlinear displacement based equations of motion are solved using GDQM and Newton-Raphson method; finally, if macro stresses converge then stresses and all other material properties would be saved for current time step, otherwise whole process repeats for current time step with new macro stresses until it converses.

## 6. RESULTS AND DISCUSSION

In this study, Graphite (T300)/Epoxy (5208) is chosen as the composite materials to analysis, in which epoxy shows nonlinear viscoelastic behaviour [6]. Prony series and calibrated nonlinear viscoelastic coefficients are used from [3].

To validate SUCM, results are compared with those obtained by MOC [3] and experiments [6]. In Fig. 2 shear stress of a 10 degree off-axis coupon versus shear strain, is shown for 0.5 minutes and 480 minutes loading. After 0.5 minutes loading, the SUCM and MOC results are very close to experimental data. As the time increases to 480 minutes, there are a little bit more differences compare to the results in the 0.5 minutes loading. Although both methods have good predictions, it seems MOC is a little bit closer to the experimental data. But it must be considered that applying SUCM is much easier than MOC.

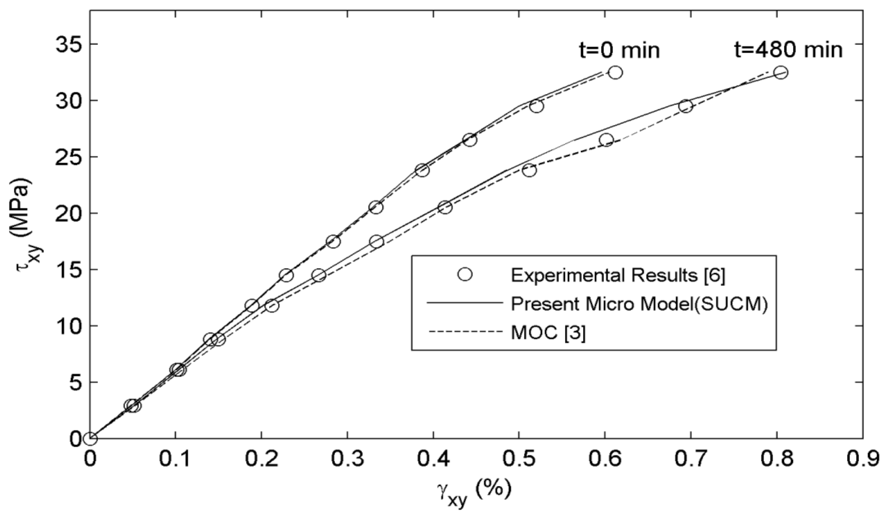


Fig. 2 Shear stress versus shear strain for 10 degree off-axis coupon.($v_f = 65\%$).

In this paper, an unsymmetrical boundary condition, FCCS (Fig. 3) is considered for nonlinear bending analysis of laminated composite plate subjected to uniform transverse load. In Fig. 4 dimensionless deflection ($\overline{w} = \frac{w}{h}$) of middle point of the free edge, versus dimensionless load ($\overline{q} = \frac{q.l_x^4}{E_2 h^4}$) of laminate with and the fiber volume fraction is shown. GDQM and FEM predictions are very close for elastic solutions (in this work, FEM is used only for validation of GDQM for elastic solutions). Increasing the applied load leads to the more deflection and also more creep behaviour of plate which is the major characterization of the nonlinear viscoelastic behaviour.

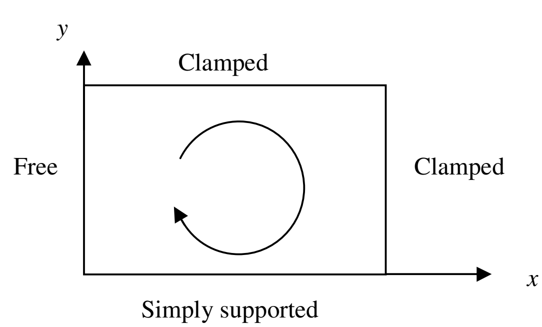


Fig. 3 FCCS boundary condition.

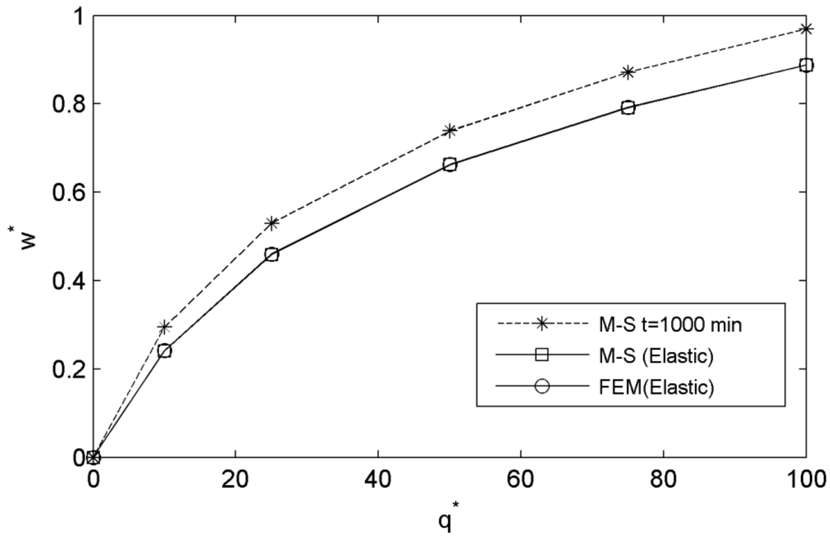


Fig. 4 Dimensionless deflection versus dimensionless applied load

( $l_y = l_x, [0/90]_s, v_f = 50\%,\ \bar{q} = 50, h = 0.03 l_x$ ).

In Fig. 5 dimensionless deflection of middle point of the free edge of laminate with versus time is shown for two different fiber volume fractions. It can be easily seen that how elastic module behaviour in micro level has direct effect on stiffness in macro level.

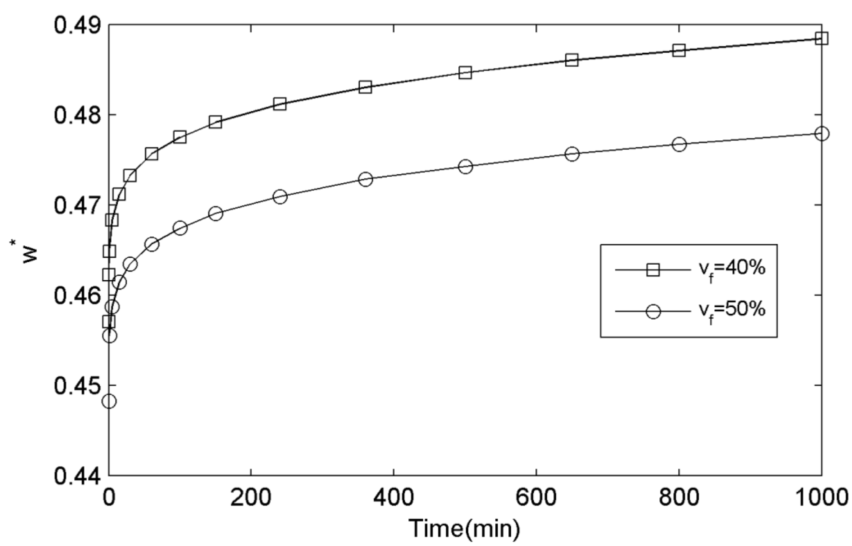


Fig. 5 Dimensionless deflection versus time for different volume fractions

( $l_y = l_x, [0/90]_s, v_f = 50\%, \bar{q} = 50, h = 0.06 l_x$ ).

## 7. CONCLUSION

Multi scale analysis of nonlinear viscoelastic laminated plates successfully applied to analysis of FCCS laminated composite plate. Owing to the stress and time dependency of materials, each point of laminate experiences different stiffness during time; so in micro scale each point need to be analyzed with different RVEs. Present multi scale method is capable of analyzing laminates with different boundary conditions, fiber volume fractions, and fiber orientations. The results in micro level have very good correlation with those obtained by experiments. GDQM as a powerful numerical method is applied to solve equations of motion of nonlinear viscoelastic laminated plate. Deflection as an important parameter for creep analysis of plates is studied by considering effects of different parameters on it. It is observed that volume fraction and load magnitude play important role on laminate's behaviour over time.

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**Appendix A**

Equation of motion in x direction after applying GDQM:

$$A_{11}(\sum_{k=1}^{N_x} a_{ik}^{(2)} u(x_k, y_j) + (\sum_{k=1}^{N_x} a_{ik} w(x_k, y_j))(\sum_{k=1}^{N_x} a_{ik}^{(2)} w(x_k, y_j))) +$$

$$A_{12}(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} v(x_k, y_l) + (\sum_{l=1}^{N_y} b_{jl} w(x_i, y_l))(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} w(x_k, y_l))) +$$

$$A_{16}(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} u(x_k, y_l) + \sum_{k=1}^{N_x} a_{ik}^{(2)} v(x_k, y_j) + (\sum_{k=1}^{N_x} a_{ik}^{(2)} w(x_k, y_j))(\sum_{l=1}^{N_y} b_{jl} w(x_i, y_l))) +$$

$$(\sum_{k=1}^{N_x} a_{ik} w(x_k, y_j))(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} w(x_k, y_l)))) + B_{11}\sum_{k=1}^{N_x} a_{ik}^{(2)} \varphi_x(x_k, y_j) +$$

$$B_{12}\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} \varphi_y(x_k, y_l) + B_{16}(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} \varphi_x(x_k, y_l) + \sum_{k=1}^{N_x} a_{ik}^{(2)} \varphi_y(x_k, y_j))) +$$

$$A_{16}(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} u(x_k, y_l) + (\sum_{k=1}^{N_x} a_{ik}^{(2)} w(x_k, y_j))(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} w(x_k, y_l))) +$$

$$A_{26}(\sum_{l=1}^{N_y} b_{jl}^{(2)} v(x_i, y_l) + (\sum_{l=1}^{N_y} b_{jl} w(x_i, y_l))(\sum_{l=1}^{N_y} b_{jl}^{(2)} w(x_i, y_l))) + A_{66}(\sum_{l=1}^{N_y} b_{jl}^{(2)} u(x_i, y_l) +$$

$$\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} v(x_k, y_l) + (\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} w(x_k, y_l))(\sum_{l=1}^{N_y} b_{jl} w(x_i, y_l)) +$$

$$(\sum_{k=1}^{N_x} a_{ik} w(x_k, y_j))(\sum_{l=1}^{N_y} b_{jl}^{(2)} w(x_i, y_l))) + B_{16}(\sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} \varphi_x(x_k, y_l)) +$$

$$B_{26}(\sum_{l=1}^{N_y} b_{jl}^{(2)} \varphi_y(x_i, y_l)) + B_{66}(\sum_{l=1}^{N_y} b_{jl}^{(2)} \varphi_x(x_i, y_l) + \sum_{l=1}^{N_y}\sum_{k=1}^{N_x} a_{ik} b_{jl} \varphi_y(x_k, y_l)) = 0$$