Three dimensional static analysis of two dimensional functionally graded plates

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ABSTRACT

In this paper, static analysis of two dimensional functionally graded plates based on three dimensional theory of elasticity is investigated. Graded finite element method has been used to solve the problem. The effects of power law exponents on static behavior of a fully clamped 2D-FGM plate have been investigated. The model has been compared with result of a 1D-FGM plate for different boundary conditions, and it shows very good agreement.

KEYWORDS

Three dimensional elasticity- Graded Finite element method- 2D-FGM plate- Static analysis

1. INTRODUCTION

Functionally graded materials (FGMs) are a new generation of advanced composite materials wherein the volume fractions of constituent materials vary continuously through the structure. Advantages of functionally graded materials (FGMs) over laminated composites are eliminating the delamination mode of failure, reducing thermal stresses, residual stresses and stress concentration factors at interfaces. The plates are widely being used in different structures and therefore, it is important to study the response of functionally graded plates under mechanical loads to optimize their resistance to failure. However investigations into static analysis for FG plates by using numerical and analytical methods are presented in Ref. [1-10]. In these papers, the material properties are assumed having a continuous variation usually in one direction. The majority of these research works deal with plate theories. A small number of investigations have been carried out on 1D-FGM clamped plates, [8-10].

Some works can be found in the literature on modeling nonhomogenous structures by using graded finite element method [11-14]. In these researches, it is shown that the conventional FEM formulation cause a discontinuous stress field in the direction perpendicular to the material property gradation, while the graded elements give a continuous and smooth variation. Therefore, by using graded finite element in which the material property is graded continuously through the elements, accuracy can be improved without refining the mesh size.

Some studies have been carried out about static, dynamic and free vibration of structures made of 2D-FGMs [15-19]. Their results indicate that the gradation of volume fractions in two directions has a higher capability to reduce the mechanical stresses and natural frequencies than conventional 1D-FGM.

Conventional functionally graded material may also not be so effective in such design problems since all outer surface of the body will have the same composition distribution. Therefore, variation of volume fraction in two directions has a higher capability to reduce the mechanical, thermal and residual stresses and leads to a flexible design than 1-D FGMs. By using graded finite element method to model the 2D-FGM plates, discontinuities and inaccuracies which are present in the conventional FEM is eliminated. Using this method, the effects of power law exponents on distribution of displacements and stresses have been investigated.

The main aim of the present paper is to present static analysis of 2D-FGM plate based on three dimensional theory of elasticity. Material properties vary through both the longitudinal and thickness directions continuously. To solve the problem graded finite element method has been applied.

2. GOVERNING EQUATIONS

2.1. Material gradient and geometry

Consider a rectangular 2D-FGM plate of length *a*, width *b* and thickness *h*, so that $0 < x \le a$, $0 < y \le b$ and $0 < z \le h$. The plate is subjected to uniform load on its top surface, while the bottom surface is free. *x*, *y* and z are the axis of Cartesian coordinate system; the *x*-axis is aligned with the longitudinal axis and the *z*-axis with the thickness direction of the plate.

Two-dimensional FGMs are usually made by continuous gradation of three or four different material phases where one or two of them are ceramics and the others are metal alloy phases, and the volume fractions of the materials vary in a predetermined composition profile. The lower surface of the plate is made of two distinct ceramics and the upper surface of two metals. c_1, c_2, m_1 and m_2 denote first ceramic, second ceramic, first metal and second metal, respectively. The volume fraction distribution function of each material can be expressed as

$$Vc_1 = \left(1 - \left(\frac{x}{a}\right)^{n_x}\right)\left(1 - \left(\frac{z}{h}\right)^{n_z}\right) \tag{1}$$

$$Vc_2 = \left(\frac{x}{a}\right)^{n_x}\left(1 - \left(\frac{z}{h}\right)^{n_z}\right) \tag{2}$$

$$Vm_1 = \left(1 - \left(\frac{x}{a}\right)^{n_x}\right)\left(\frac{z}{h}\right)^{n_z} \tag{3}$$

$$Vm_2 = \left(\frac{x}{a}\right)^{n_x}\left(\frac{z}{h}\right)^{n_z} \tag{4}$$

where n_x and n_z are non-negative volume fraction exponents through the *x* and *z* directions.

Material properties at each point can be obtained by using the linear rule of mixtures; therefore the material property *P* such as modulus of elasticity and mass density in the 2D-FGM plate is determined by linear combination of volume fractions and material properties of the basic materials as [13]

$$P = Pc_1 Vc_1 + Pc_2 Vc_2 + Pm_1 Vm_1 + Pm_2 Vm_2 \tag{5}$$

The volume fractions in Eqs. (1)- (4) reduce to the conventional 1D-FGMs for $n_x = 0$ and in this case the material properties vary only through the thickness direction, for this case the lower surface of plate is made of a ceramic and the upper surface of a metal alloy. The basic constituents of the 2D-FGM plate are presented in Table 1.

It should be noted that Poisson's ratio is assumed to be constant through the body. This assumption is reasonable because of the small differences between the Poisson's ratios of basic materials.

2.2. Equilibrium equations

In the absence of body forces, the equilibrium equations for 2D-FGM rectangular plates can be written as follows:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \tag{6}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \tag{7}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{8}$$

where *u*, *v*, and *w* are the displacement components along the *x*, *y* and *z* axes, respectively.

2.3. Stress-strain relations

The stress- strain relations of linear elasticity from the Hook's law in terms of the modulus of elasticity *E* and Poisson's ratio ν in matrix form are as follow

$$[\sigma] = [D][\varepsilon] \tag{9}$$

$$[\sigma] = \{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx}\}^{\mathrm{T}} \tag{10}$$

$$[\varepsilon] = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{yz} \quad \varepsilon_{zx}\}^{\mathrm{T}} \tag{11}$$

$$[D]=\frac{E(x,z)(1-\nu)}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{pmatrix} \tag{12}$$

It should be noted that E varies in the x and z directions and ν is assumed to be constant.

2.4 Strain-displacement relations

The strain displacement relations of the infinitesimal theory of elasticity in the rectangular Cartesian coordinates are as

$$[\varepsilon]=[d][q] \tag{13}$$

Calling the mathematical operator [d] by

$$[d]=\begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 1/2\frac{\partial}{\partial y} & 1/2\frac{\partial}{\partial x} & 0 \\ 0 & 1/2\frac{\partial}{\partial z} & 1/2\frac{\partial}{\partial y} \\ 1/2\frac{\partial}{\partial z} & 0 & 1/2\frac{\partial}{\partial x} \end{pmatrix} \tag{14}$$

$$[q]=\begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{15}$$

2.4. Boundary conditions

For all-round clamped plate, the essential boundary conditions are as

$$u,v,w\ (x,0,z)=u,v,w\ (x,b,z)=u,v,w\ (a,y,z)=u,v,w\ (0,y,z)=0 \tag{16}$$

The stress boundary conditions which should be satisfied during solution are as

$$\begin{gathered}\sigma_{yz}(x,y,0)=\sigma_{yz}(x,y,h)=\sigma_{xz}(x,y,0)=\sigma_{xz}(x,y,h)=\sigma_{zz}(x,y,0)=0,\\ \sigma_{zz}(x,y,h)=p_z\end{gathered} \tag{17}$$

3. GRADED FINITE ELEMENT MODELING

Consider a three dimensional 8-node linear brick shape element in the rectangular Cartesian coordinates. Nodal coordinates are known in the global *xyz*- coordinates. Using the finite element approximation to the displacement field, the displacement component are approximated by shape function *N*, as

$$[q]^{(e)}=[N]^{(e)}[\delta]^{(e)} \tag{18}$$

$$[\delta]^{(e)}=\{U_1\ \ V_1\ \ W_1\ \ .\ \ .\ \ .\ \ U_8\ \ V_8\ \ W_8\}^{\mathbf{T}} \tag{19}$$

where $[\delta]^{(e)}$ is the nodal displacement matrix.

$$[N]^{(e)}=\begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 \end{bmatrix} \tag{20}$$

where $N_i, i=1,2,\ldots,8$ are the shape functions of 8-node linear brick element. The components of matrix $[N]^{(e)}$ are as

$$N_i(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta) \tag{21}$$

where $(-1\le\xi\le1),(-1\le\eta\le1)$ and $(-1\le\zeta\le1)$.

To model the problem, Graded Finite Element Method is used. To treat the material inhomogeneity in an FGM, we can use graded elements, which incorporate the material property gradient at the size scale of the element. The graded elements employ the same shape functions to interpolate the unknown displacements, the geometry, and the material parameters. This approach effectively represents the material variation at the element level and results in smooth solution transition across the element boundaries. Using the graded elements in modeling of gradation of the material properties results in a manner that is more accurate than dividing the solution domain into homogenous elements

$$P = \sum_{i=1}^{8} P_i N_i \tag{22}$$

where P_i is the material property corresponding to node *i*.

Substituting Eq. (18) in Eq. (13) gives the strain matrix of element (e) as

$$[\varepsilon]^{(e)} = [d][N]^{(e)} [\delta]^{(e)} \tag{23}$$

$$[\varepsilon]^{(e)} = [B]^{(e)} [\delta]^{(e)} \tag{24}$$

where $[B]^{(e)} = [d][N]^{(e)}$ [20].

The finite element model can be derived using Rayleigh Ritz energy formulation. The details of this method could be found in different textbooks [20]. By applying this method to the governing equations, the stiffness and force element matrices in Cartesian coordinate system are as follows:

$$[K]^{(e)} = \int_{V_{(e)}} [B]^T [D][B] dV \tag{25}$$

$$[F]^{(e)} = \int_{A_{(e)}} [N]^T \{T\} dA \tag{26}$$

where

$$\{\mathrm{T}\} = \begin{Bmatrix} 0 \\ 0 \\ p_z \end{Bmatrix} \tag{27}$$

As the plate is subjected to a uniform load on its top surface, *x* and *y* components of force matrice are equal to zero. To evaluate stiffness matrix using 8-point Guass quadrature rule, we used a transformation between Cartesian coordinate system into local coordinates system ($-1 \le \xi, \eta, \zeta \le 1$) [20].

Now by assembling the element matrices, the global equilibrium equations for the 2D-FGM plate can be obtained as

$$[K]\{\delta\} = \{F\} \tag{29}$$

4. RESULTS AND DISCUSSIONS

4.1 Verification

The present solution can be verified using data of a 1D-FGM plate under the same loading that were previously presented in [8]. Therefore, the parameters are given as $n_x = 0$, $n_z = 1$, $a = b, \frac{h}{a} = 0.2, E_{c2} = 70 GPa$, $E_{m2} = 200 GPa$, $P = 1 Pa$ and $\upsilon = 0.3$. The non-dimensional transverse displacement through the thickness for a 1D-FGM plate with different boundary conditions is considered here, and the present results are compared with the published data. Fig. 1 shows good agreement between these results.

4.2. Numerical results

4.2.1. Static analysis

Consider a 2D-FGM square plate with a side-length *a=b=1m* and nondimensional thickness *h/a*=0.4. The plate is subjected to a uniform static load on its top surface. Constituent materials are two distinct ceramics and two distinct metals described in Table 1. The static pressure and the Poisson's ratio are taken as constant values: $P = 40 MPa$ and $\upsilon = 0.3$. The number of graded elements through the x,y and z directions are 17*17*7.

Figs. 2 shows the distribution of transverse displacement *w* for a clamped plate on different surfaces at $y = \frac{a}{2}$ for the power law exponents $n_x = n_z = 1$. This figure shows that the maximum magnitude of transverse displacement belongs to the upper surface and its value decreases through the thickness.

Figs. 3, 4 and 5 show the distribution of in-plane displacements *u* and *v* and transverse displacement *w* for a clamped plate at $z = \frac{h}{2}, y = \frac{a}{2}$ for different values of power law exponents. Fig. 3 denotes that in-plane displacement *u* is antisymmetrical about *x=0.5 m* for $n_x = 0, n_z = 2$. It means, if the longitudinal power law exponent be equal to zero and the material properties vary only through the thickness direction, the distribution of in plane displacement *u* should be antisymmetrical. Results illustrate that by increasing the axial power law exponent n_z, the magnitude of *u* and *v* is decreased, this fact can be seen in Figs. 3 and 4.

Figs. 6 and 7 show the variation of out-of-plane normal stress σ_z and out-of-plane shear stress σ_{xz} with the power law exponents for a clamped plate through the thickness at centerline $x = y = \frac{a}{2}$ and $x = y = \frac{a}{4}$, respectively. Fig. 6 illustrates that the distribution of σ_z through the thickness of 2D-FGM clamped plate is almost equal for different values of power law exponents. Also, the natural boundary conditions at upper and lower surfaces of the plate are satisfied, this fact can be seen in Figs. 6 and 7. As it can be seen from the results, the distribution of stresses has continuous variations due to using graded elements.

5. CONCLUSIONS

Static analysis of two dimensional functionally graded plates based on three dimensional theory of elasticity is considered. Material properties vary through both longitudinal and thickness directions. The Graded Finite Element Method, and Rayleigh-Ritz energy formulation is applied. The proposed method is verified by the result of a 1D-FGM plate under the same loading which is extracted from published literature. The comparisons between the results show that the present method has a good compatibility with the existing results. The effects of power law exponents on the behavior of clamped plates are investigated. The present results represent that mechanical stress distribution can be modified to a required manner by selecting an appropriate volume fraction profiles in two directions and this gives designers a powerful tool for flexible designing of structures under multifunctional requirements. Also results demonstrate that using graded elements provide smoother and more accurate results than homogeneous elements to model the inhomogeneous structures.

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Table 1 Basic constituent of 2D-FGM plate [13]

Constituent	Material	*E (GPa)*
m_1	Ti6Al4V	115
m_2	Al 1,100	69
c_1	SiC	440
c_2	Al2O3	300

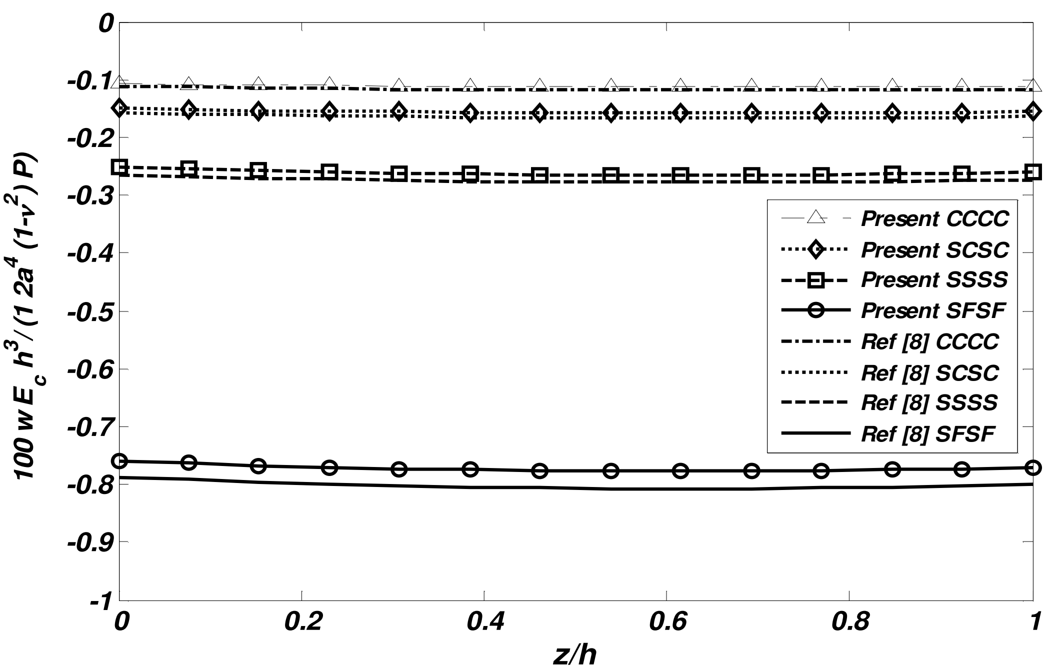

Fig. 1 Non-dimensional transverse displacement through the thickness at $x = y = \frac{a}{2}$ compared with Ref [8]

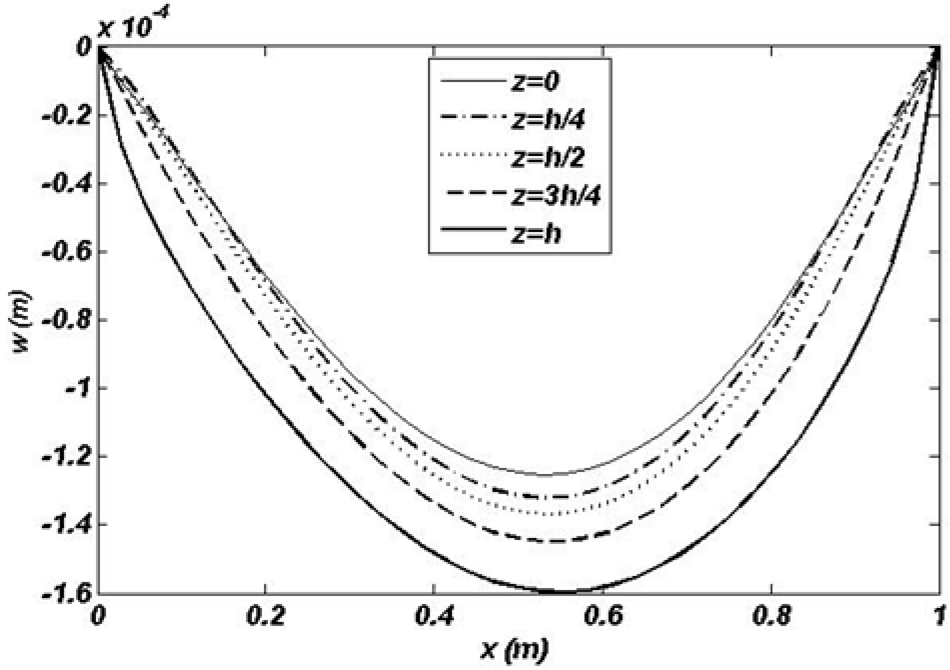

Fig. 2 Transverse displacement w on different surfaces for a clamped plate at $y = \frac{a}{2}$ for $n_x = n_z = 1$

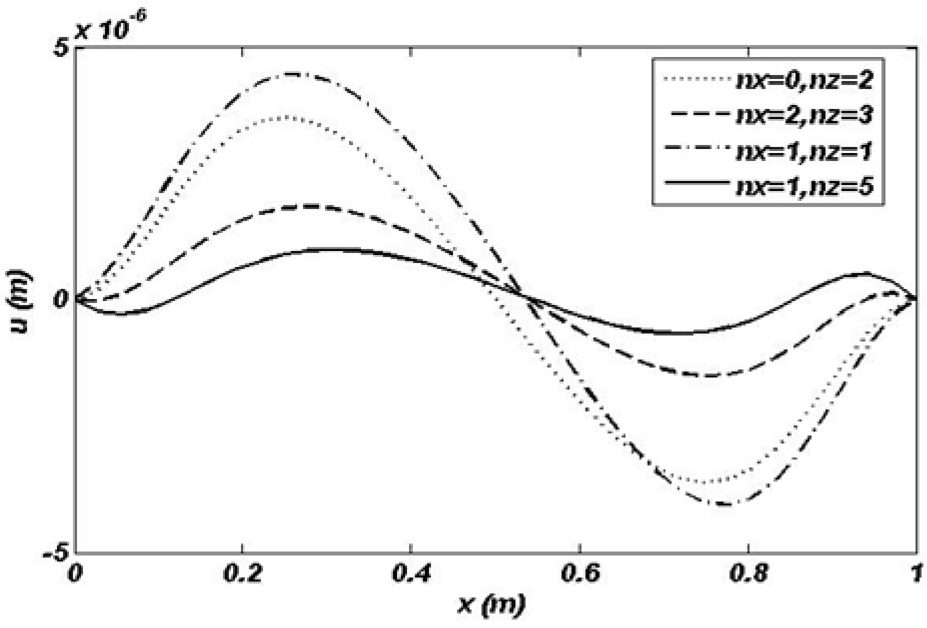

Fig. 3 In-plane displacement u for a clamped plate at $z = \frac{h}{2}, y = \frac{a}{2}$ for different power law exponents

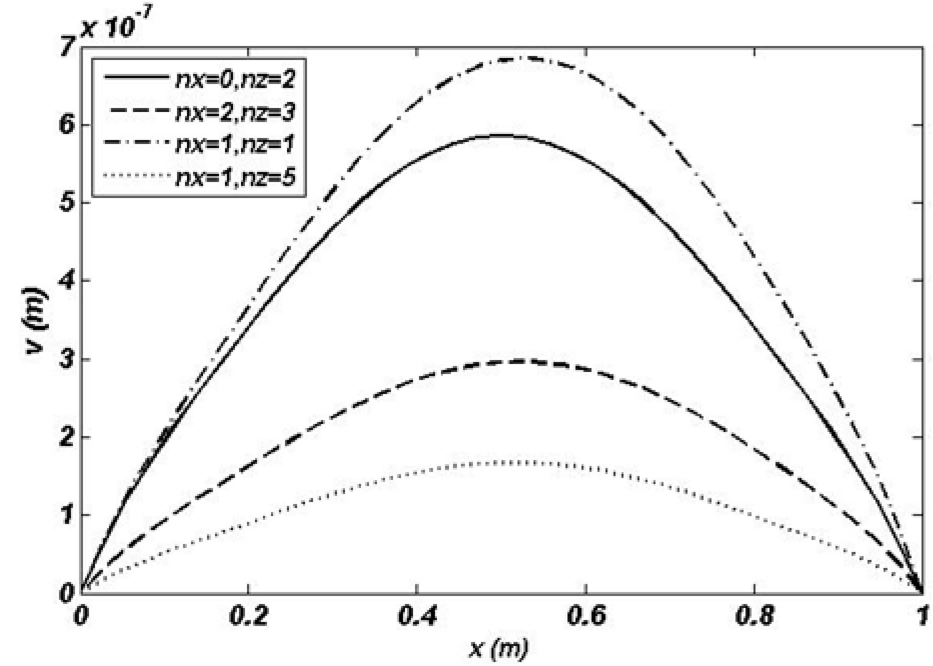

Fig. 4 In-plane displacement v for a clamped plate at $z = \frac{h}{2}, y = \frac{a}{2}$ for different power law exponents

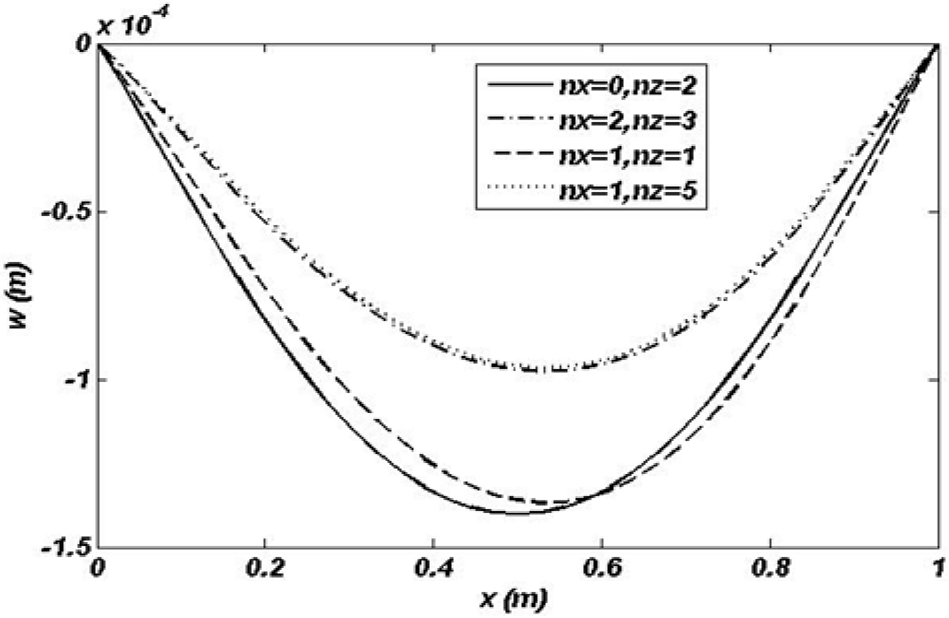

Fig. 5 Transverse displacement w for a clamped plate at $z = \frac{h}{2}, y = \frac{a}{2}$ for different power law exponents

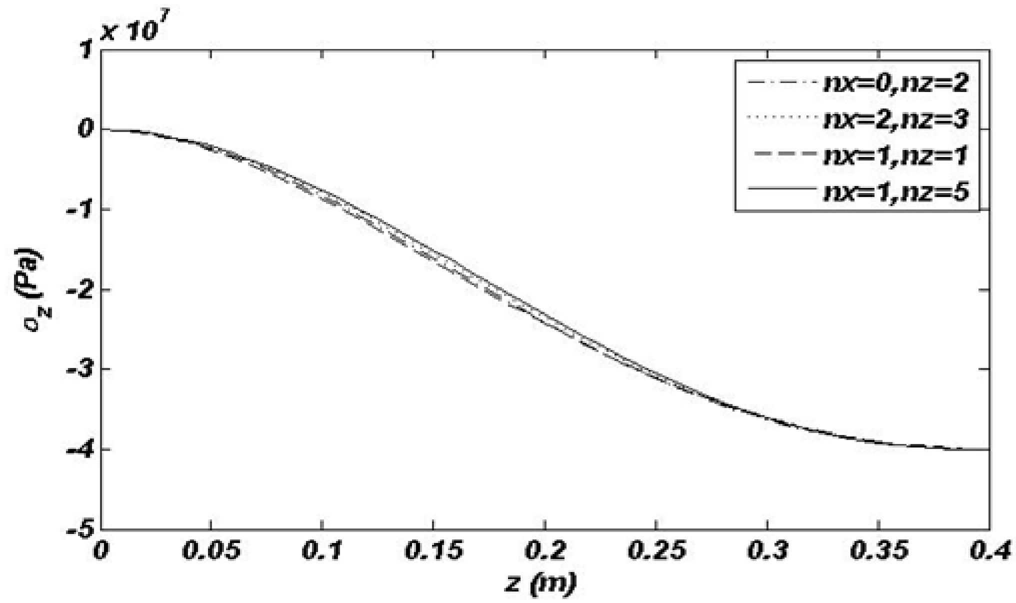

Fig. 6 Out-of-plane normal stress $\boldsymbol{\sigma}_z$ for a clamped plate through the thickness at $\boldsymbol{x} = \boldsymbol{y} = \frac{\boldsymbol{a}}{2}$ for different power law exponents

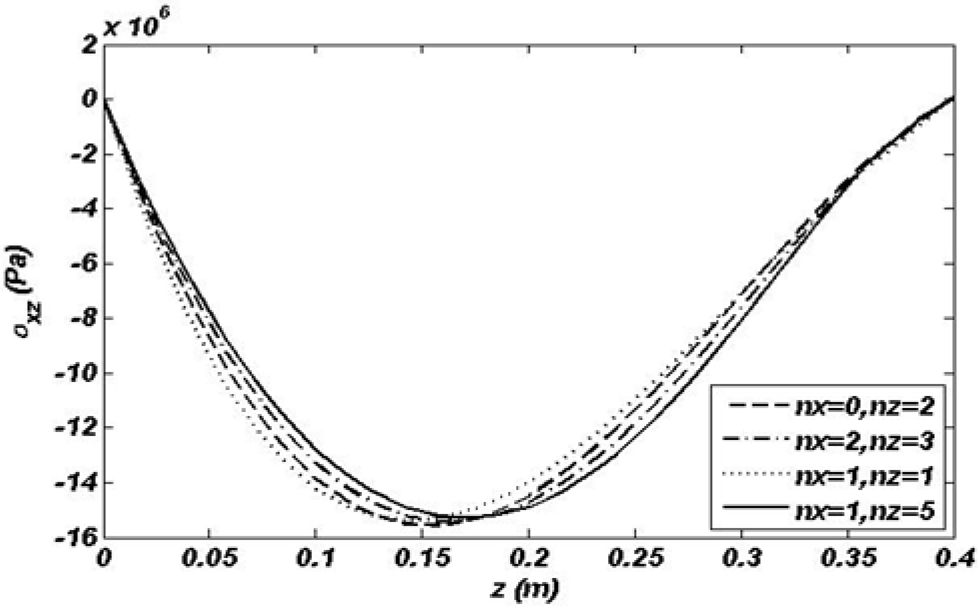

Fig. 7 Out-of-plane shear stress $\boldsymbol{\sigma}_{xz}$ for a clamped plate through the thickness at $\boldsymbol{x} = \boldsymbol{y} = \frac{\boldsymbol{a}}{4}$ for different power law exponents