EVALUATING TWO-DIMENSIONAL WARRANTY RESERVE WITH ACCOUNTING FOR USAGE INTENSITY

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ABSTRACT

Evaluating warranty reserve fund is one of the most intricate problems undertaken by sellers. Many researchers introduced two dimensional models to build realistic models for evaluating warranty cost as a function of product age and its usage extent within a specified warranty period. Most researchers assumed that customers have similar attitudes with respect to usage. However, such similar attitudes cannot be guaranteed. The present work proposes a new random parameter γ in the failure rate model in order to account for the random variation of usage intensity and its effect on the product characteristic life. Expected value of warranty reserve was calculated using mathematical formulas. Furthermore, Monte Carlo simulation was used to calculate warranty reserve for different policies at different risk probabilities.

KEYWORDS

warranty reserve, two-dimensional warranty, usage intensity

1. INTRODUCTION

The abundance of research work of the investigation and study of the determination of warranty reserve along an extended period of more than 50 years is clearly noticed. This could be traced back to the works of Menke [1], Blischke and Murthy [2]–[5]. The developed Models for this problem could be categorized as regards to the supplier's responsibility to:

- Replacement of non-repairable components upon failure by new ones [6]–[8].
- Repair of failed components where the effect of imperfection of repair is considered [9]–[11].

While warranty data in general offers a wide variety of information that can be used in reliability studies[12]–[14]. As regards to failure probability density function, the models can be categorized to:

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- One dimensional models that account only for the warranty span of time with no regard to the usage intensity [9], [15], [16]. These models are built on the implicit assumption that all customers have the same behaviour as regards to usage of the product [15], [17], [18].

- Two Dimensional models aiming that take into consideration the variation of usage intensity from one customer to another. Warranty agreements pre-specify upper limit for the usage as well as for warranty period in order to protect suppliers from severely high usage rates [19]. The two dimensions are the usage limit U and the warranty period W as the time limit [20]–[22]. One of the first trials in the two dimensional models is the work of Eliashberg et al [6]. In their work, the usage as a time dependent covariate is expressed in the form of logistic function. They have tried to model the effect of usage on the failure rate of the product in the form of additive model. A conditional failure density function is thus obtained. D.K. Manna et al. [23] have developed an integral equation for the two dimensional renewal function . They have stated that it is impossible to obtain an analytical expression for the renewal function.

Time and usage that differs from a customer to another are critical factors in determining the warranty reserve policies and costs. For this reason, in this paper a new formulation of the problem is introduced to cover time and usage while considering uncertainty of usage intensity in evaluating warranty reserve. Monte Carlo simulation is used to calculate warranty reserve for different warranty policies at different risk probabilities. It is also used to validate the mathematical model developed. Simulation results give same results of the mathematical model when calculating the expected warranty reserve.

The paper is organized as follows: section 2 represents the joint probability distribution of time to failure and usage intensity, a new parameter γ is demonstrated that represents the usage intensity and its effect on the failure density function. Section 3 demonstrates the mathematical elaboration to calculate the expected warranty reserve for different warranty policies: mixed warranty policy, full rebate warranty policy, and linear prorata warranty policy. Section 4 represents the evaluation of the expected warranty policy using Monte Carlo simulation, with respect to usage intensity for different warranty policies. Section 5 discusses a given case study with numerical assumptions that give a comparison for different warranty policies between the two approaches mathematical and simulation, under one-dimensional and two-dimensional warranty policies. Section 6 explains the calculation warranty reserve under specified risk probabilities. Finally, conclusions are presented in section 7.

2. A JOINT PROBABILITY DISTRIBUTION OF TIME TO FAILURE AND USAGE INTENSITY

For some categories of products such as automobiles, warranty agreements usually prescribe a specified usage upper limit UL beyond which the warranty coverage does no longer exist. Two parameter warranty (UL, W) models are applied in this case. Even three parameters warranty models are known in case of airplanes (warranty period, kilometres of flight and number of take-offs). The reason behind developing multi-parameter warranty models is essentially to protect sellers and producers against the fact that failure rates of products are generally not only function of time but also function of usage intensity.

An approach is proposed to the problem of the two-dimensional warranty model as follows: The Characteristic life η of any product, subjected to normal or design load, is defined as that period of time during which only one failure could be expected. This is given by the following formula:

$$\int_0^{\eta} h(t)dt = 1 \tag{1}$$

Where, $h(t)$ is the failure or hazard rate of the product.

As the loading intensity increases/decreases, the characteristic life, as that period of time during which one failure could be expected, will decrease/increase accordingly. If η is the characteristic life in case of normal usage, then by introducing a new parameter γ accounting for usage intensity, the characteristic life will be assumed η / γ. Therefore,

$$\int_0^{\eta/\gamma} h(t)dt = 1 \tag{2}$$

The failure rate could be expressed as follows:

$$h(t) = a\, t^{m} \tag{3}$$

After integration equation 4 is obtained as follows:

$$\frac{a(\eta/\gamma)^{m+1}}{m+1} = 1 \quad putting \;\; m+1 = \beta \;\; then \quad a = \frac{\beta}{(\eta/\gamma)^{\beta}}$$

Therefore,

$$h(t) = \frac{\beta}{(\eta/\gamma)^{\beta}}\; t^{\beta-1} \tag{4}$$

This formula is firstly given for the case of normal (design) loading intensity (γ=1) by Weibull failure probability could be accordingly obtained as follows:

$$F(t|\gamma) = 1 - e^{-\int h(t)dt} = 1 - e^{-\left(\frac{\gamma t}{\eta}\right)^{\beta}} \tag{5}$$

Hence, failure probability density function is obtained as follows:

$$f(t|\gamma) = \frac{dF(t)}{dt} = \gamma^{\beta}\,\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\; e^{-\left(\frac{\gamma t}{\eta}\right)^{\beta}} \tag{6}$$

Where, β, η are shape and scale parameters of the distribution.

In Fig.(1) a system of failure density functions with different values of usage intensity parameter γ is presented. When the usage intensity γ increases the failure rate f(t) increases and the characteristic life η decreases. In Fig.(2), the justification of introducing parameter γ is presented. On normal usage, the usage limit (U_L) and time limit (W) are reached simultaneously (γ = 1). Light usage (U) is characterized by the event of reaching the time limit before reaching the usage limit (γ < 1). Heavy usage (γ > 1) is the case of reaching usage limit earlier than the time limit [24], [25].

Therefore, the usage intensity parameter γ could be expressed as follows:

$$\gamma = \left(U/U_L\right) / \left(T_W/W\right) \tag{7}$$

In case of having, $\gamma \leq 1$, $T_W = W$ $\;then$

$$\gamma = U/U_L \tag{8}$$

On the other hand, in case of having, $\gamma > 1$, $\;U = U_L$ $\;then$

$$\gamma = {}^{W}\!/_{T_W} \tag{9}$$

Where, T_W is the time of early termination of the warranty coverage in the case of reaching the upper usage limit prescribed in the warranty agreement before completing the warranty period.

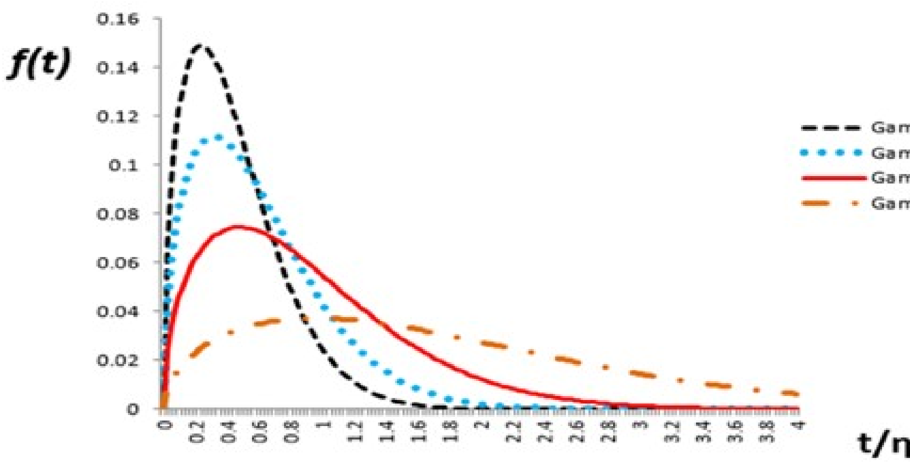

Fig.1 Effect of Usage Severity on Failure Density Function

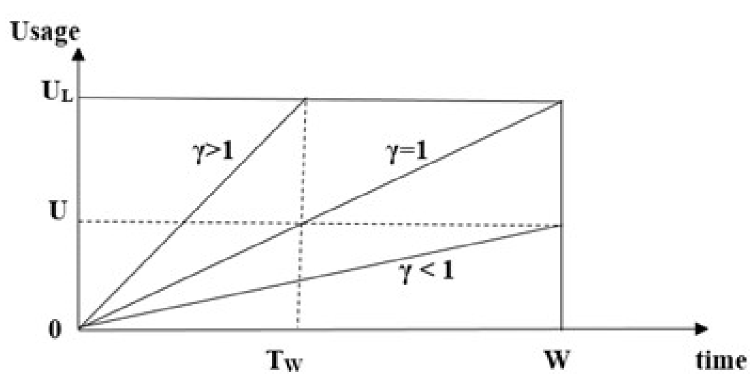

Fig. 2 Warranty Period for different values of Usage Intensity.

Usage intensity γ varies randomly from customer to customer. Hence, γ should be considered as a random variable. Data on γ could be collected from statistics of completed two-dimensional warranty agreements. Since the maximum value of γ is unlimited whereas minimum value tends to zero, Weibull distribution could be considered eligible to describe the randomness of usage intensity parameter γ because of its universality provided by its shape parameter. Therefore, probability density function of γ is taken as a truncated Weibull distribution as follows:

$$f(\gamma) = \left[\frac{1}{e^{-\left(\frac{\gamma_{min}}{\zeta}\right)^{\alpha}} - e^{-\left(\frac{\gamma_{max}}{\zeta}\right)^{\alpha}}} \right] * \frac{\alpha}{\zeta} \left(\frac{\gamma}{\zeta}\right)^{\alpha-1} e^{-\left(\frac{\gamma}{\zeta}\right)^{\alpha}} \quad (10)$$

Where,

γ_{min} ,γ_{max} are minimum and maximum values of γ

α, ζ are shape and scale parameters of the distribution

Having (6) and (10), the Joint Probability Density Function of time to failure and usage parameter γ will be expressed as follows:

$f(t, \gamma) = f(t|\gamma) * f(\gamma)$

$$f(t,\gamma) = \left[\frac{1}{e^{-\left(\frac{\gamma_{min}}{\zeta}\right)^{\alpha}} - e^{-\left(\frac{\gamma_{max}}{\zeta}\right)^{\alpha}}} \right] * \left(\frac{\alpha}{\zeta}\right) \left(\frac{\gamma\beta}{\eta}\right) \left(\frac{\gamma}{\zeta}\right)^{\alpha-1} \left(\frac{\gamma t}{\eta}\right)^{\beta-1} e^{-\left[\left(\frac{\gamma t}{\eta}\right)^{\beta} + \left(\frac{\gamma}{\zeta}\right)^{\alpha}\right]} \quad (11)$$

3. Expected Value of Warranty Reserve For Different Warranty Policies

There are three different ways considered for a warranty policy: a- full rebate, b- Prorata, and c- mixed. Consider a lot of products of size L. The unit selling price C including warranty cost. Equation (12) represents the cost warranty function. It is required to evaluate the expected value of warranty reserve necessary and sufficient to undertake the warranty liabilities along a specified warranty period. As it is clear from Fig.3, during the period from date of delivery up to a period w_1, a full rebate policy is adopted, whereas from w_1 up to the end of warranty period at w_2 a pro rata policy is adopted.

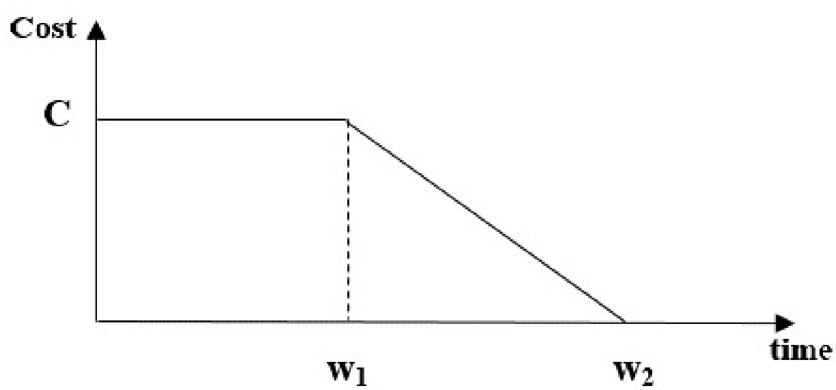

Fig.3 Mixed Warranty Policy

$$C(t) = \begin{cases} C\,, & 0 \leq t \leq w_1 \\ C * \left(1 - \frac{t - w_1}{w_2 - w_1}\right), & w_1 \leq t \leq w_2 \end{cases} \quad (12)$$

The element of warranty reserve cost (WR) incurred is evaluated as follows:

$$dWR = C(t) * L * f(t,\gamma)\,dt\,d\gamma \quad (13)$$

The expected value of the warranty reserve ratio (WRR) per unit product will be expressed as follows:

$$\overline{WRR} = \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_0^{W_1'} f(t|\gamma)dt\right]d\gamma + \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_{W_1'}^{W_2'} \left(1 - \frac{t - W_1'}{W_2' - W_1'}\right) f(t|\gamma)dt\right]d\gamma \quad (14)$$

$$\overline{WRR} = \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_0^{W_2'} f(t|\gamma)dt\right]d\gamma - \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_{W_1'}^{W_2'} \left(\frac{t - W_1'}{W_2' - W_1'}\right) f(t|\gamma)dt\right]d\gamma \quad (15)$$

$$W_m' = min\left(W_m, \frac{W_2}{\gamma}\right) \qquad m = 1,2 \quad (16)$$

Where W_m' equals to $\dfrac{W_2}{\gamma}$ is the time of early termination of the warranty agreement in case of heavy usage ($\gamma > 1$)

After some manipulations and treatments as given in the Appendix and by applying Simpson's rule, the integrals in equation (15) could be reduced to forms amenable to practical computations. Thereby the expected value of the warranty reserve ratio $\overline{WRR}$ will take the following form:

$$\overline{WRR} = 1 - Q\sum_{j=1}^{M} K_j\left(A_j + \eta * B_j - E_j\right) \quad (17)$$

Where,

$$Q = \frac{\alpha\,(\gamma_{max} - \gamma_{min})}{\zeta^{\alpha}[3*(M-1)]*\left[e^{-\left(\frac{\gamma_{min}}{\zeta}\right)^{\alpha}} - e^{-\left(\frac{\gamma_{max}}{\zeta}\right)^{\alpha}}\right]} \quad (18)$$

(M - 1) is the number of intervals through which the range of integration (0 to 1), as indicated in the Appendix, is subdivided to be able to apply Simpson's rule. It should be an even integer number. The accuracy of computation increases as *M* increases.

$K_1 = K_M = 1 \quad K_j = 2 \;\; if\; j\; is\; even\; number\; otherwise\;\; K_j = 4 \;\; j = 2,3,\ldots,M-1$

$$A_j = \gamma_j^{\alpha-1} e^{-\left[\left(\frac{\gamma_j}{\zeta}\right)^{\alpha} + \left(\frac{\gamma_{j*} W'_{2j}}{\eta}\right)^{\beta}\right]} \quad (19)$$

$$W'_{mj} = min\left(W_m, \frac{W_2}{\gamma_j}\right) \qquad m = 1,2 \quad (21)$$

$$B_j = \frac{\gamma_j^{\alpha-2} e^{-\left(\frac{\gamma_j}{\zeta}\right)^{\alpha}}}{\left(W'_{2j} - W'_{1j}\right)} \left[\mathrm{G}\left[\left(\frac{\gamma_j W'_{2j}}{\eta}\right)^{\beta}, \frac{1}{\beta}\right] - \mathrm{G}\left[\left(\frac{\gamma_j W'_{1j}}{\eta}\right)^{\beta}, \frac{1}{\beta}\right] \right] \quad (22)$$

$$E_j = \left(\frac{W'_{1j}}{W'_{2j} - W'_{1j}}\right) * \gamma_j^{\alpha-1} e^{-\left(\frac{\gamma_j}{\zeta}\right)^{\alpha}} \left[e^{-\left(\frac{\gamma_j W'_{1j}}{\eta}\right)^{\beta}} - e^{-\left(\frac{\gamma_j W'_{2j}}{\eta}\right)^{\beta}} \right] \quad (23)$$

The other two policies: full rebate policy and pro rata policy, warranty reserve is obtained from expression (17) as special cases as follows:

Full Rebate Policy ($W_1 = W_2$),

In this case, that $B_j = 0$ and $E_j = 0$, therefore,

$$\overline{WRR} = 1 - Q \sum_{j=1}^{M} K_j \; A_j \quad (24)$$

Pro Rata Policy ($W_1 = 0$) then:

$E_j = 0$ and B_j is reduced to D_j as follows:

$$D_j = \frac{\gamma_j^{\alpha-2} e^{-\left(\frac{\gamma_j}{\zeta}\right)^{\alpha}}}{W'_{j2}} \; G\left[\left(\frac{\gamma_j W'_{j2}}{\eta}\right)^{\beta}, \frac{1}{\beta}\right] \quad (25)$$

$$\overline{WRR} = 1 - Q \sum_{j=1}^{M} K_j \left(A_j + \eta D_j \right) \quad (26)$$

4. Evaluation of Warranty Reserve Using Monte Carlo Simulation

Fig.4 shows the steps of running the Monte Carlo simulation to calculate the warranty reserve. Evaluation of warranty reserve by Monte Carlo simulation provides three important advantages:

1. Validating of the mathematical expressions obtained in section 3.

2. Evaluating warranty reserves ratio at different risk probabilities. The risky situation is the case of having warranty expenditures exceeds that allocated funds as warranty reserve. It should be noted that mathematical expressions yield only expected value or the mean of warranty reserve ratio.

3. Getting samples of data of warranty reserve sufficient to elaborate a test for goodness of fit in order to decide the appropriate distribution of warranty reserve ratio as a random variable.

5. CASE STUDY

Given a product with the following characteristics $Shape\ factor\ \beta = 2$

$Scale\ factor\ (charactristic\ life)\quad \eta = 12\ months$

The usage intensity parameter γ is taken as a random variable distributed according to Weibull distribution with parameters:

$shape\ factor\ \alpha = 3\ and\ scale\ factor\ \zeta = 2\ \ \gamma_{min} = 0.8\ \ \gamma_{max} = 3$

The following will be calculated:

1) Evaluate the expected two-dimensional warranty reserve ratio for the following warranty policies:

a) Full Rebate warranty policy for a period of W_2 = 10 months

b) Pro Rata for the period of W_2 = 10 months

c) Mixed policy with W_1 taking different values: 7, 5 and 3 months whereas W_2=10 months

2) Evaluate the one dimensional warranty reserve *ratio* for the three different policies by keeping the warranty period unchanged for different usage intensities.

Results of calculation of warranty reserve ratio are given in Table (1).

The remarkable coincidence between results of calculations by the expressions (17), (24) and (26) and results of Monte Carlo simulation is clearly noticed in Table-1. The warranty reserve ratio $\overline{\text{WRR}}$ value is almost the same for mathematical and Monte Carlo simulation for all policies. As already stated before, the reason behind developing multi-parameter warranty models is essential to protect sellers and producers against the fact that failure rates of products are generally not only function of time but also function of usage intensity. To demonstrate this fact, one-dimensional warranty is evaluated to be compared with that of two dimensional warranty. In one-dimensional warranty, periods of warranty are constant irrespective of the increase in usage intensity. Therefore, considerable increase in $\overline{\text{WRR}}$ as compared with that of two-dimensional warranty as shown in Table-1.

Table-1 Comparison between two dimensional and one dimensional warranty for the expected values of warranty reserve Ratio $\overline{\mathrm{WRR}}$

	Warranty Reserve Ratio $\overline{\mathrm{WRR}}$				
	Full Rebate	Pro Rata	Mixed		
W1	10	0	7	5	3
W2	10	10	10	10	10
	Two Dimensional Warranty				
Simulation	0.497	0.189	0.481	0.440	0.339
Mathematical	0.498	0.19	0.482	0.442	0.342
	One Dimensional Warranty				
Simulation	0.833	0.415	0.747	0.664	0.563
Mathematical	0.833	0.417	0.752	0.668	0.567

6. Evaluation of Warranty Reserve Ratio For Different Risk Probabilities

Application of Monte Carlo simulation and getting samples of sufficient size for the warranty reserve ratio gave the possibility of deciding an appropriate distribution for the warranty reserve ratio as a random variable. Having the distribution of WRR enables researchers and engineers to evaluate the warranty reserve funds with a predetermined risk probability of the event of having warranty expenditures exceed allocated funds.

Application of normality plot test reveals the clear goodness of fit of the sample of WRR to the normal distribution. Therefore, the warranty reserve ratio that should be allocated to keep risk probability down to a specified value ($0 < \lambda < 1$) is given as follows:

$$WRR(\lambda) = \overline{WRR} + \sigma \ast \Phi^{-1}\{\Phi(Z_{min}) + (1-\lambda) \ast [\Phi(Z_{max}) - \Phi(Z_{min})]\} \tag{27}$$

Where, $Z = \frac{WRR - \overline{WRR}}{\sigma}$

$\overline{WRR}$, σ, WRR_{max}, WRR_{min} are the mean, standard deviation, maximum and minimum values of WRR as obtained from simulation. $\Phi(Z)$ Standard normal distributions

The Full rebate warranty policy requires the highest WRR while the Prorata policy requires the minimum. Furthermore, the value of WRR depends upon the duration of the Full rebate in the mixed warranty policy.

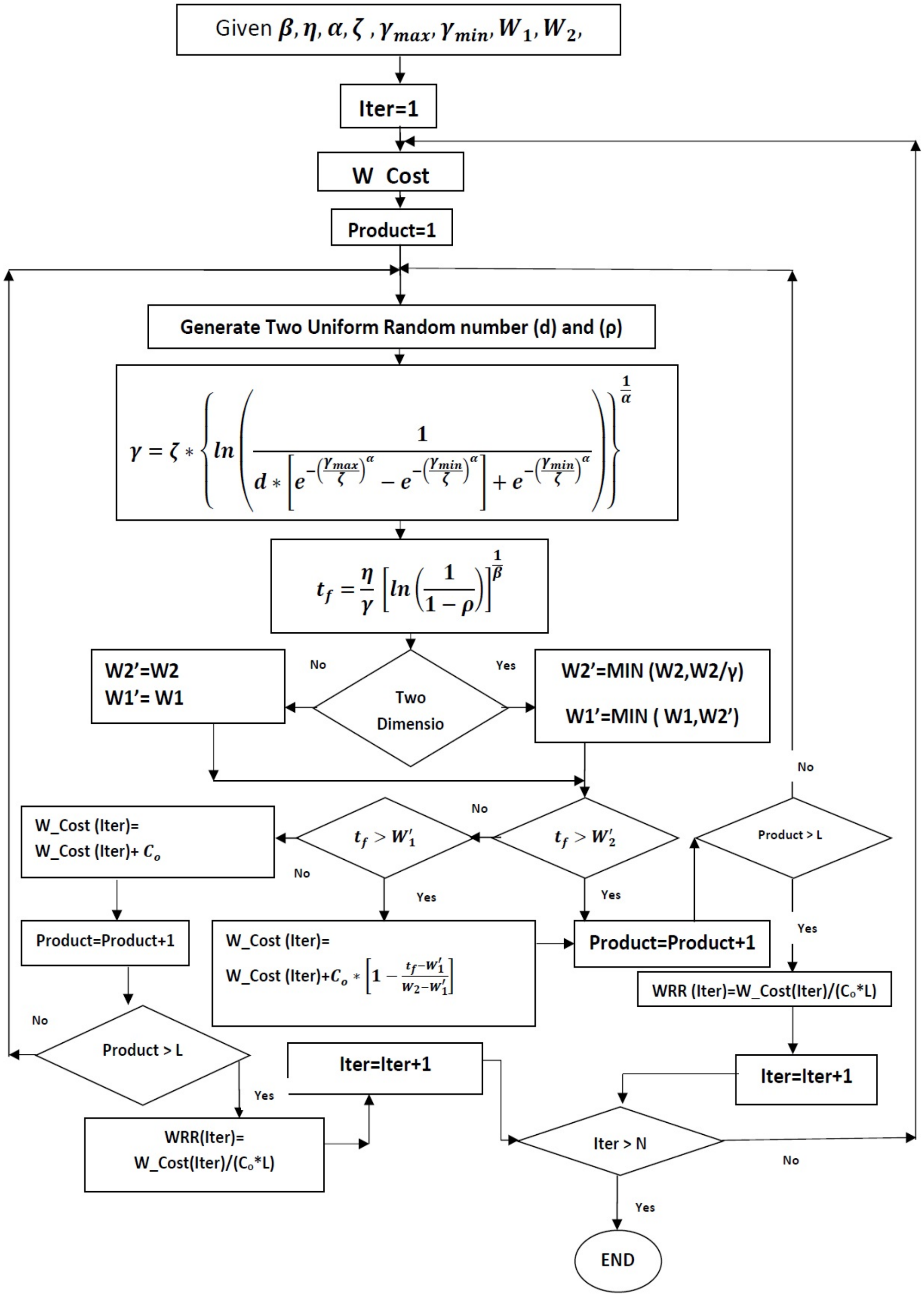

Fig. 4. Flowchart of Monte Carlo Simulation calculating WRR

7. CONCLUSIONS

The following conclusions can be obtained from this study:

1) Consideration of usage intensities in warranty agreements is mandatory to protect sellers and producers against cases of severe usages of products. This is clearly demonstrated in the comparisons presented in Table 1.

2) The newly introduced usage intensity parameter Υ as a random variable enables users not only to obtain expressions amenable to practical computations but also to build a Monte Carlo simulation model for the purpose of their validation.

3) The randomness of warranty expenditures is resulting from the randomness of both failure events and the randomness of usage intensities. The coincidence between the cumulative observed relative frequencies, as obtained from Monte Carlo simulation, with that of standard normal distribution clearly supports the claim that warranty expenditure is a random variable distributed normally.

4) Warranty reserves allocated on the basis of expected value only may lead to the risk of having warranty expenditures exceeding those allocated reserves with probability may reach to 50%. Expression (27) of the present work enables sellers and producers to evaluate warranty reserves with predetermined risk probabilities based on the proved normality of warranty expenditures.

Finally, Warranty reserve problems in case of having systems with different structures, of single product, and accounting for the imperfection of repair in evaluating these reserves are recommended for future research.

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APPENDIX

Consider the integrals in (15):

$$\overline{WRR} = \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_0^{W_2'} f(t|\gamma)dt\right]d\gamma - \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_{W_1'}^{W_2'}\left(\frac{t-W'_1}{W'_2-W'_1}\right)f(t|\gamma)dt\right]d\gamma$$

$$\overline{WRR} = I_1 - I_2$$

$$I_1 = \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_0^{W_2'} f(t|\gamma)dt\right]d\gamma$$

But from (6), We get

$$f(t|\gamma)dt == \gamma^{\beta}\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{\gamma t}{\eta}\right)^{\beta}} dt = -d\left[e^{-\left(\frac{\gamma t}{\eta}\right)^{\beta}}\right]$$

$$I_1 = -\int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_0^{W_2'} d\left[e^{-\left(\frac{\gamma t}{\eta}\right)^{\beta}}\right]\right]d\gamma = \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[1-e^{-\left(\frac{\gamma * W_2'}{\eta}\right)^{\beta}}\right]d\gamma$$

$$I_1 = \int_{\gamma_{min}}^{\gamma_{min}} f(\gamma)d\gamma - \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\, e^{-\left(\frac{\gamma * W_2'}{\eta}\right)^{\beta}} d\gamma = 1 - \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\, e^{-\left(\frac{\gamma * W_2'}{\eta}\right)^{\beta}} d\gamma$$

Considering (10) in the above integral, we get:

The integral I_1 will be reduced to a form amenable to numerical integration as follows:

$$Put \quad y = \frac{\gamma - \gamma_{min}}{\gamma_{max} - \gamma_{min}} \quad \gamma = y(\gamma_{max} - \gamma_{min}) + \gamma_{min} \quad then$$

$$I_1 = 1 - \frac{\alpha(\gamma_{max} - \gamma_{min})}{\zeta^\alpha\left[e^{-\left(\frac{\gamma_{min}}{\zeta}\right)^\alpha} - e^{-\left(\frac{\gamma_{max}}{\zeta}\right)^\alpha}\right]} \int_0^1 \gamma^{\alpha-1} e^{-\left(\frac{\gamma}{\zeta}\right)^\alpha} e^{-\left(\frac{\gamma * W'_2}{\eta}\right)^\beta} dy$$

Applying Simpson's rule of numerical integration, we find:

$$I_1 = 1 - Q \sum_{j=1}^{M} K_j A_j \tag{1A}$$

Where,

$$Q = \frac{\alpha(\gamma_{max} - \gamma_{min})}{\zeta^\alpha[3 * (M-1)] * \left[e^{-\left(\frac{\gamma_{min}}{\zeta}\right)^\alpha} - e^{-\left(\frac{\gamma_{max}}{\zeta}\right)^\alpha}\right]}$$

$K_1 = K_M = 1 \quad K_j = 4$ if j is even number otherwise $K_j = 2$

$$A_j = \gamma_j^{\alpha-1} e^{-\left[\left(\frac{\gamma_j}{\zeta}\right)^\alpha + \left(\frac{\gamma_j * W'_{2j}}{\eta}\right)^\beta\right]} \quad W'_{2j} = min\left(W_2, \frac{W_2}{\gamma_j}\right) \quad \gamma_j = \frac{j-1}{M-1}(\gamma_{max} - \gamma_{min}) + \gamma_{min}$$

The second integral I_2 will be evaluated as follows:

$$I_2 = \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_{W'_1}^{W'_2}\left(\frac{t - W'_1}{W'_2 - W'_1}\right) f(t|\gamma) dt\right] d\gamma = I_3 - I_4$$

$$I_3 = \int_{\gamma_{min}}^{\gamma_{max}} f(\gamma)\left[\int_{W'_1}^{W'_2}\left(\frac{t}{W'_2 - W'_1}\right) \gamma \frac{\beta}{\eta}\left(\frac{\gamma t}{\eta}\right)^{\beta-1} e^{-\left(\frac{\gamma t}{\eta}\right)^\beta} dt\right] d\gamma$$

Put

$$\left(\frac{\gamma t}{\eta}\right)^\beta = x \qquad \frac{\gamma t}{\eta} = x^{\frac{1}{\beta}} \qquad t = \frac{\eta}{\gamma} x^{\frac{1}{\beta}}$$

$$I_3 = \int_{\gamma_{min}}^{\gamma_{max}} \frac{f(\gamma)}{W'_2 - W'_1}\left(\frac{\eta}{\gamma}\right)\left[\int_{\left(\frac{\gamma * W'_1}{\eta}\right)^\beta}^{\left(\frac{\gamma * W'_2}{\eta}\right)^\beta} x^{\frac{1}{\beta}} e^{-x} dx\right] d\gamma$$

$$I_3 = \int_{\gamma_{min}}^{\gamma_{max}} \frac{f(\gamma)}{W'_2 - W'_1}\left(\frac{\eta}{\gamma}\right)\left[\int_0^{\left(\frac{\gamma * W'_2}{\eta}\right)^\beta} x^{\frac{1}{\beta}} e^{-x} dx - \int_0^{\left(\frac{\gamma * W'_1}{\eta}\right)^\beta} x^{\frac{1}{\beta}} e^{-x} dx\right] d\gamma$$

The integral

$$\int_0^{\left(\frac{\gamma * W_1'}{\eta}\right)^\beta} x^{\frac{1}{\beta}} e^{-x} dx = G\left[\left(\frac{\gamma * W_1'}{\eta}\right)^\beta, \frac{1}{\beta}\right] = \left(\frac{\gamma * W_1'}{\eta}\right)^{\beta+1} \left[\frac{1}{3(M-1)}\right] \sum_{m=1}^{M} K_m H_m$$

$$H_m = y_m^{\frac{1}{\beta}} e^{-y_m\left(\frac{\gamma * W_1'}{\eta}\right)^\beta} \qquad y_m = \frac{m-1}{M-1} \qquad m = 1,2,\ldots,M$$

is known as the lower gamma function $G\left[\left(\frac{\gamma * W_1'}{\eta}\right)^\beta, \frac{1}{\beta}\right]$

$$I_3 = \int_{\gamma_{min}}^{\gamma_{max}} \frac{f(\gamma)}{W'_2 - W'_1}\left(\frac{\eta}{\gamma}\right)\left[G\left[\left(\frac{\gamma W_2'}{\eta}\right)^\beta, \frac{1}{\beta}\right] - G\left[\left(\frac{\gamma W_1'}{\eta}\right)^\beta, \frac{1}{\beta}\right]\right] d\gamma$$

Consider expression (10) and change variable of integration:

$$y = \frac{\gamma - \gamma_{min}}{\gamma_{max} - \gamma_{min}} \qquad \gamma = y(\gamma_{max} - \gamma_{min}) + \gamma_{min} \qquad d\gamma = (\gamma_{max} - \gamma_{min})dy$$

$$I_3 = \frac{\alpha\eta(\gamma_{max} - \gamma_{min})}{\zeta^\alpha\left[e^{-\left(\frac{\gamma_{min}}{\zeta}\right)^\alpha} - e^{-\left(\frac{\gamma_{max}}{\zeta}\right)^\alpha}\right]}\int_0^1 \frac{\gamma^{\alpha-2} e^{-\left(\frac{\gamma}{\zeta}\right)^\alpha}}{W'_2 - W'_1}\left[G\left[\left(\frac{\gamma W_2'}{\eta}\right)^\beta, \frac{1}{\beta}\right] - G\left[\left(\frac{\gamma W_1'}{\eta}\right)^\beta, \frac{1}{\beta}\right]\right] dy$$

The integral is evaluated numerically by Simpson's rule as stated before.

$$I_3 = \eta * Q \textstyle\sum_{j=1}^{M} K_j * B_j \qquad (2A)$$

$$B_j = \frac{\gamma_j^{\alpha-2} e^{-\left(\frac{\gamma_j}{\zeta}\right)^\alpha}}{(W_{j2}' - W_{j1}')}\left[G\left[\left(\frac{\gamma_j W_{2j}'}{\eta}\right)^\beta, \frac{1}{\beta}\right] - G\left[\left(\frac{\gamma_j W_{1j}'}{\eta}\right)^\beta, \frac{1}{\beta}\right]\right]$$

$$G_j\left[\left(\frac{\gamma W_{lj}'}{\eta}\right)^\beta, \frac{1}{\beta}\right] = \left(\frac{\gamma W_{lj}'}{\eta}\right)^{\beta+1} * \left[\frac{1}{3*(N-1)}\right]\sum_{m=1}^{N} K_m * H_{mlj} \qquad l = 1,2$$

$$H_{mj} = z_m^{\frac{1}{\beta}} e^{-z_m\left(\frac{\gamma W_j'}{\eta}\right)^\beta}$$

$$z_m = \frac{i-1}{N-1} \qquad (i = 1,2,\ldots,N)$$

$$I_4 = \int_{\gamma_{min}}^{\gamma_{max}} \left(\frac{W'_1}{W'_2 - W'_1}\right) f(\gamma)\left[\int_{W_1'}^{W_2'} \gamma \frac{\beta}{\eta}\left(\frac{\gamma t}{\eta}\right)^{\beta-1} e^{-\left(\frac{\gamma t}{\eta}\right)^\beta} dt\right] d\gamma$$

Following the same procedure as done for evaluation of I_1, we find:

$$I_4 = Q \sum_{j=1}^{M} K_j \, E_j \quad (3A)$$

Where, $E_j = \left(\frac{W'_{1j}}{W'_{2j} - W'_{1j}}\right) * \gamma_j^{\alpha-1} e^{-\left(\frac{\gamma_j}{\xi}\right)^{\alpha}} \left[e^{-\left(\frac{\gamma_j W'_{1j}}{\eta}\right)^{\beta}} - e^{-\left(\frac{\gamma_j W'_{2j}}{\eta}\right)^{\beta}} \right]$

Therefore, from (2A) and (3A) we find for integral I_2 the following expression:

$$I_2 = Q \sum_{j=1}^{M} K_j * \left(\eta * B_j - E_j\right) \quad (4A)$$

From (1A) and (4A), the expected warranty reserve ratio in case of *mixed policy* will take the following form:

$$\overline{WRR} = 1 - Q \sum_{j=1}^{M} K_j \left(A_j + \eta * B_j - E_j\right) \quad (5A)$$