A CLASS OF BOUNDARY VALUE METHODS FOR THE COMPLEX DELAY DIFFERENTIAL EQUATION

Shifeng Wu

Department of Computer Science, Guangdong Polytechnic Normal University, Guangzhou, 510665, China

ABSTRACT

In this paper, a class of boundary value methods (BVMs) for delay differential equations (DDEs) is considered. The delay dependent stable regions of the extended trapezoidal rules of second kind (ETR2s), which are a class of BVMs, are displayed for the test equation of DDEs. Furthermore, it is showed ETR2s cannot preserve the delay-dependent stability of the complex coefficient test equation considered. Some numerical experiments are given to confirm the theoretical results.

AMS 2000 Mathematics Subject Classification: 65L20, 65M12

KEYWORDS

Delay differential equations; boundary value methods; delay-dependent stability; extended trapezoidal rules of second kind

1. INTRODUCTION

The stability of numerical methods plays an important role in the numerical solution of initial value problems (IVPs). During the past decade, most of the work on the asymptotic stability for delay differential equations (DDEs) dealt with finding the stability region independently of the delay term. Compared with the delay-independent analysis, the stability analysis for a fixed value of the delay is much more difficult [1, 2, 3, 4]. In recent years, some studies have been devoted to delay-dependent stability (see, for example, Zhao [5], J. Ma [6], Aceto[7, 8], Abdelhameed[9]).

It is less known that the control of the parasitic solutions is much easier if the problem is transformed into an almost equivalent boundary value problem. Starting from such an idea, a new class of multistep methods, called boundary value methods (BVMs), has been proposed and analyzed in the last few years(see, for example, Amodio and Mazzia [10], Brugnano ,Iavernaro and Trigiante [11], Brugnano and Trigiante[12,13], Brugnano [14]). BVMs no longer suffer the limitations the Dahlquist barriers [15,16,17] and the impossibility to define stable, high-order symplectic methods, but also permits us to avoid the difficulties when changing the stepsize. Consequently, A-stable, essentially sympletic BVMs of any order are obtained. The extended trapezoidal rules of second kind (ETR_2s), which are a class of BVMs, are very important. In fact, they result to be the methods to be chosen when approximating either continuous boundary value problem [13] or Hamiltonian problems [10,13]. In particular, they are all "essentially" symplectic methods [14].

---

*Supported by the Foundation for Distinguished Young Talents in Higher Education of Guangdong, China (Grant No. 2014KQNCX175).

†Corresponding: fengtree@126.com

In this paper, we investigate the delay-dependent stability of ETR_2s, which are a class of BVMs, for the test equation of complex DDEs. This paper is organized as follows. In section 2, the stability concepts and definitions are introduction for the coefficient test equation. In section 3, we deal with the ETR_2s for the complex coefficient version of equation. Finally, in section 4, several numerical experiments are given to confirm the theoretical results presented in previous sections.

2. THE ETR2S FOR DDES

Consider the test equation for DDEs

$$\begin{cases} y'(t) = \bar{a}y(t) + \bar{b}y(t-\tau), t > 0, \\ y(t) = g(t), t \in [-\tau, 0], \end{cases} \quad (2.1)$$

where $\tau > 0$, the constants $\bar{a}, \bar{b} \in \mathbb{C}$ and $g(t)$ is a continuous function. Since we are interested in analyzing equation (2.1) for an arbitrary but fixed delay, we remark that, by using a scaling of the time variable, we are able to bring equation (2.1) into the from

$$\begin{cases} y'(t) = ay(t) + by(t-1), t > 0, \\ y(t) = g(t), t \in [-1, 0], \end{cases} \quad (2.2)$$

where $a := \bar{a}\tau$ and $b := \bar{b}\tau$. Therefore, there is no loss of generality in performing the stability analysis for equation (2.2).

We consider the following ETR_2s [13,14], which are a class of boundary value methods, with $(v, v-1)$-boundary conditions. For $y'(t) = f(t, y), y(0) = y_0$, ETR_2s have the following general form:

$$\sum_{i=0}^{v-1} \alpha_i (y_{n-v+i} - y_{n+v-1-i}) = \frac{h}{2}(f_n + f_{n-1}), n = v, \cdots, N-1, \quad (2.3)$$

and values

$$y_0, \cdots, y_{v-1}, \ \ y_N, \cdots, y_{N+v-2} \ \ fixed,$$

are needed. A set of additional equations is needed. Such equation can be derived by additional methods having the same order as the main methods, and the stability properties of the global method are inherited by the main formula (2.3). For more detail on the practical use of the additional method, see [13].

The coefficients $\{\alpha_i\}$ of (2.3) are determined so that the maximum possible order $p = K + 1$ is reached. In Table 1, the normalized coefficients $\hat{\alpha_i} = \alpha_i \eta_i$ of these methods, up to $K = 9$, are displayed [13,14]. Also in this case, for $v = 1$, one obtains the based trapezoidal rule.

Table 3.1. Coefficients of ETR_2s

K	v	η_k	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$
1	1	1	-1				
3	2	12	-1	-9			
5	3	120	1	-15	-80		
7	4	840	-1	14	-126	-525	
9	5	5040	1	-15	120	-840	-3024

We apply the ETR$_2$s (2.3) with constant stepsize $h = 1/m$ to (2.2), with m a positives integer. This leads to

$$\sum_{i=0}^{v-1} \alpha_i (y_{n-v+i} - y_{n+v-1-i}) = \frac{h}{2} a(y_n + y_{n-1}) + \frac{h}{2} b(y_{n-m} + y_{n-m-1}), \quad (2.4)$$

for $n = v, \cdots, N-1$ and $y_{-m}, \cdots, y_{v-1}, y_N, \cdots, y_{N+v-2}$, are needed. where $y_i = g(t_i)$ for $i \leq 0$ and $y_1, \cdots, y_{v-1}, y_N, \cdots, y_{N+v-2}$ can be be derived by additional methods

Next, the characteristic equation of (2.4) is given by

$$\begin{aligned} P_{v,m}(z; a, b) := \sum_{i=0}^{v-1} \alpha_i (z^{n-v+i} - z^{n+v-1-i}) - \\ \frac{h}{2} a(z^n + z^{n-1}) - \frac{h}{2} b(z^{n-m} + z^{n-m-1}) = 0, \end{aligned} \quad (2.5)$$

where $P_m(z; a, b)$ is called the stability polynomial. In order to research the stability of the numerical solution of (2.4), the following proposition is given [13].

Proposition 2.1 The following statements are equivalent:

(1)$P_{v,m}$ has $m + v - 1$ zeros inside $|z| = 1$ and 1 zeros outside $|z| = 1$,

(2) the discrete numerical solution y_n of (2.4), with constant stepsize $h = 1/m$, satisfies $\lim_{n \to \infty} y_n = 0$ for all initial functions $g(t)$.

Lemma 2.2 For the $q(v, \varphi) = \frac{\rho(v,\varphi)}{1+cos\ \varphi}$,where

$\rho(v, \varphi) = \sin\ \varphi \sum_{i=0}^{v-1} \alpha_i \left(2 \sum_{j=0}^{v-1-i} \cos\ j\varphi - 1\right), (v = 1, \cdots, 5)$, then

(i) the function $q^2(v, \varphi)$ is strictly monotonically increasing for $\varphi \in (0, \pi)$ when $v < 6$;

(ii)the $q(v, \varphi) < q(v+1, \varphi), (v = 1, \cdots, 4)$ when $\varphi \in (0, \pi)$.

Proof. Firstly, we prove (i).

For $v = 1, \varphi \in (0, \pi)$, $q^2(v, \varphi)$ changes to

$$q(1, \varphi) = -\frac{sin\ \varphi}{1 + cos\ \varphi} = -\tan\ \frac{\varphi}{2}, \quad (2.6)$$

$$\frac{dq(1, \varphi)}{d\varphi} = -\frac{1}{1 + cos\ \varphi} \leq -\frac{1}{2}. \quad (2.7)$$

The $q(1, \varphi)$ is negative and strictly monotonically decreasing function with respect to φ, so $q^2(1, \varphi)$ is strictly monotonically increasing function.

For $v = 2, \varphi \in (0, \pi)$, $q^2(v, \varphi)$ changes to

$$q(2, \varphi) = \frac{1}{12} \frac{sin\ \varphi}{1 + cos\ \varphi} (-1(2\cos\ \varphi + 1) - 9), \quad (2.8)$$

$$\frac{dq(2, \varphi)}{d\varphi} = -\frac{1}{6} \frac{cos^2\varphi + cos\ \varphi + 4}{1 + cos\ \varphi} \leq -\frac{1}{2}. \quad (2.9)$$

It shows that the $q(2, \varphi)$ is negative and strictly monotonically decreasing function with respect to φ, so $q^2(2, \varphi)$ is strictly monotonically increasing function.

For $v=3, \varphi \in (0,\pi)$, $q^2(v,\varphi)$ changes to

$$\begin{aligned} q(3,\varphi) &= \frac{1}{120}\frac{\sin\varphi}{1+\cos\varphi}((2(\cos 2\varphi+\cos\varphi)+1)-15(2\cos\varphi+1)-80) \\ &= \frac{1}{30}\frac{\sin\varphi}{1+\cos\varphi}(\cos^2\varphi-7\cos\varphi-24), \end{aligned} \tag{2.10}$$

$$\frac{dq(3,\varphi)}{d\varphi} = -\frac{1}{30}\frac{-2\cos^3\varphi+6\cos^2\varphi+9\cos\varphi+17}{1+\cos\varphi} \le -\frac{1}{2}. \tag{2.11}$$

A straightforward computation of (2.10) shows that the $q(3,\varphi)$ is negative and strictly monotonically decreasing function with respect to φ, so $q^2(3,\varphi)$ is strictly monotonically increasing function.

For $v=4, \varphi \in (0,\pi)$, $q^2(v,\varphi)$ changes to

$$\begin{aligned} q(4,\varphi) &= \frac{1}{840}\frac{\sin\varphi}{1+\cos\varphi}[-(2(\cos 3\varphi+\cos 2\varphi+\cos\varphi)+1) \\ &\quad +14(2(\cos 2\varphi+\cos\varphi)+1)-126(2\cos\varphi+1)-525] \\ &= \frac{1}{210}\frac{\sin\varphi}{1+\cos\varphi}[-2\cos^3\varphi+13\cos^2\varphi-55\cos\varphi-166], \end{aligned} \tag{2.12}$$

$$\frac{dq(4,\varphi)}{d\varphi} = -\frac{1}{210}\frac{6\cos^4\varphi-24\cos^3\varphi+36\cos^2\varphi+81\cos\varphi+111}{1+\cos\varphi} \le -\frac{1}{2}. \tag{2.13}$$

A straightforward computation of (2.12) shows that the $q(4,\varphi)$ is negative and strictly monotonically decreasing function with respect to φ, so $q^2(4,\varphi)$ is strictly monotonically increasing function.

For $v=5, \varphi \in (0,\pi)$, $q^2(v,\varphi)$ changes to

$$\begin{aligned} q(5,\varphi) &= \frac{1}{5040}\frac{\sin\varphi}{1+\cos\varphi}[(2(\cos 4\varphi+\cos 3\varphi+\cos 2\varphi+\cos\varphi)+1) \\ &\quad -15(2(\cos 3\varphi+\cos 2\varphi+\cos\varphi)+1) \\ &\quad +120(2(\cos 2\varphi+\cos\varphi)+1)-840(2\cos\varphi+1)-3024] \\ &= \frac{1}{630}\frac{\sin\varphi}{1+\cos\varphi}[2\cos^4\varphi-14\cos^3\varphi+51\cos^2\varphi-173\cos\varphi-496]. \end{aligned} \tag{2.14}$$

$$\frac{dq(5,\varphi)}{d\varphi} = -\frac{1}{630}\frac{-8\cos^5\varphi+40\cos^4\varphi-80\cos^3\varphi+80\cos^2\varphi+275\cos\varphi+323}{1+\cos\varphi} \le -\frac{1}{2}. \tag{2.15}$$

A straightforward computation of (2.14) shows that the $q(5,\varphi)$ is negative and strictly monotonically decreasing function, so $q^2(5,\varphi)$ is strictly monotonically increasing function. Therefore, the $q(v,\varphi)$ are negative and strictly monotonically decreasing function.

Secondly, from (2.6)-(2.15), for $\varphi \in (0,\pi)$

$$\frac{dq(1,\varphi)-dq(2,\varphi)}{d\varphi} = \frac{1}{6}\frac{\cos^2\varphi+\cos\varphi-2}{1+\cos\varphi} < 0, \tag{2.16}$$

$$\frac{dq(2,\varphi)-dq(3,\varphi)}{d\varphi} = \frac{1}{30}\frac{-2\cos^3\varphi+\cos^2\varphi+4\cos\varphi-3}{1+\cos\varphi} < 0, \tag{2.17}$$

$$\frac{dq(3,\varphi)-dq(4,\varphi)}{d\varphi}=\frac{1}{210}\frac{6\cos^4\varphi-10\cos^3\varphi-6\cos^2\varphi+18\cos\ \varphi-8}{1+\cos\ \varphi}<0, \quad (2.18)$$

$$\frac{dq(4,\varphi)-dq(5,\varphi)}{d\varphi}=\frac{1}{630}\frac{-8\cos^5\varphi+22\cos^4\varphi-8\cos^3\varphi-28\cos^2\varphi+32\cos\ \varphi-10}{1+\cos\ \varphi}<0. \quad (2.19)$$

From (2.16)-(2.19) and $q(v,0)=0(v=1,\cdots,5)$, we get

$$q(v,\varphi)<q(v+1,\varphi),(v=1,\cdots,4). \quad (2.20)$$

So, we get the Lemma2.2.

3. The Complex Coefficient Case

In this section the stability definitions given in Section 2 are extended to the complex coefficient version of the test equation (2.2). Hence, we assume in the continuation$a,b,g(t)\in\mathbb{C}$. We denote by Ξ_* the stability region of the equation (2.2), that is the set of complex pairs (a,b) such that $\lim_{t\to\infty}|y(t)|=0$ for all initial function $g(t)$. Now, for the numerical solution, the following definitions are given by Guglielmi [3] and Bell [1].

Definition 3.1 The τ-stability region (D-stability region) of a numerical method for DDEs is the set

$$Q_\tau=\bigcap_{m>1}Q_m,$$

where, for a given positive integer m , Q_m is the set of the pair of complex numbers (a,b) such that the discrete numerical solution $\{y_n\}_{n\geq 0}$ of (2.2), with constant stepsize $h=1/m$ and m a positive integer, satisfies $\lim_{n\to\infty}|y_n|=0$ for all initial functions $g(t)$.

Definition 3.2 The numerical method for DDEs is τ-stable (D-stable) if $Q_\tau\supseteq\Xi_*$.

3.1. The true stability region

To analyze the stability region of (2.2), we consider the stability set $\Xi_*[a]$ (with $a\in\mathbb{C}$ and fixed) and represent it in the b-plane. The parametric equations of the boundary of the stability region (cf. [1]) are

$$V_*[a]:=\{(\Re[b_*],\Im[b_*])\in\mathbb{R}^2|\theta\in\mathbb{R}:\mathrm{K}(exp(i\theta);a,b_*)=0\},$$

where

$$\begin{cases}\Re[b_*](\theta;a)=c_\rho(\theta;a)-\theta\sin\ \theta,\\ \Im[b_*](\theta;a)=c_i(\theta;a)+\theta\cos\ \theta,\end{cases} \quad (3.1)$$

with

$$\begin{cases}c_\rho(\theta;a)=-\Re(a)\cos\ \theta+\Im(a)\sin\ \theta,\\ c_i(\theta;a)=-\Im(a)\cos\ \theta-\Re(a)\sin\ \theta,\end{cases} \quad (3.2)$$

for the parameter θ varies in $(-\infty,+\infty)$. Observe that the boundary is the ßum" of a circle centered in the origin with the radius $|a|$ and of the transcendental curve $(-\theta\sin\ \theta,\theta\sin\ \theta)$, which does not depend on a.

The form of the stability region in the complex b-plane is reported in Fig 3.1 for a given complex value of $a=-1+i$.

3.2. Extended analysis of ETR_2s for the complex DDEs

We focus attention on the complex coefficient version of the characteristic polynomial (2.5) and study the corresponding boundary locus, which we denote here by $V_m(v)$, we denote by $V_{m,a}(v)$ the restriction of the boundary locus of the $ETR_2(v,m)$ obtained by fixing a and m, we denote by $Q_\tau(v), Q_\tau[a](v), Q_m(v)$ and $Q_m[a](v)$ the previously defined Q_τ and Q_m for the special case of the $ETR_2(v,m)$ or fixed $a \in \mathbb{C}$. The straightforward discussion involves the explicit determination of $V_{m,a}(v)$. By standard algebraic manipulation we obtain

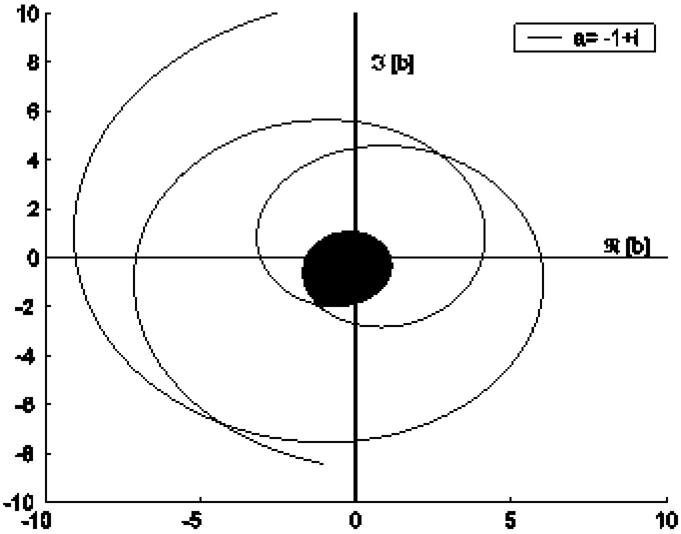

Figure 3.1: An example of the stability set $\Xi_*[a]$ (shaded region) for the complex test equation (2.2), with a section of the boundary curve.

$$V_{m,a}(v) := \{(\Re[b_m](\theta; a, v), \Im[b_m](\theta; a, v)) \in R^2 | \theta \in (-m\pi, m\pi)\},$$

where

$$\begin{cases} \Re[b_m](\theta; a, v) = c_\rho(\theta; a) + 2mq(v, \frac{\theta}{m}) \sin\ \theta, \\ \Im[b_m](\theta; a, v) = c_i(\theta; a) - 2mq(v, \frac{\theta}{m}) \cos\ \theta, \end{cases} \tag{3.3}$$

with $c_\rho(\theta; a)$ and $c_i(\theta; a)$ given by (3.2), $q(v, \frac{\theta}{m})$ given by $q(v, \varphi) = \frac{\rho(v,\varphi)}{1+\cos\ \varphi}$, $\rho(v, \varphi) = \sin\ \varphi \sum_{i=0}^{v-1} \alpha_i \left(2 \sum_{j=0}^{v-1-i} \cos\ j\varphi - 1\right)$. We observe that some interesting relationships persist between the two sets $V_*[a]$ and $V_{m,a}(v)$. In fact, by comparing (3.3) and (3.1), we see that the numerical stability boundary is still given by the 'sum' of the circle (3.1) and of a regular curve which does not depend on a. However, the all of $ETR_2(v,m), (v = 1, \cdots, 5)$ are unfortunately not τ-stable. We have the following Theorem.

Theorem 3.3 The $ETR_2(v,m)$ (2.4)($v = 1, \cdots, 5$) is not τ-stable .

In order to prove this Theorem 3.3, we need the following Lemma 3.4.

Lemma 3.4 For fixed a and the $ETR_2(v,m)$, ($v = 1, \cdots, 5$), such that $\Re[a] < 0$ and $\Im[a] \neq 0$. Then

$$\Xi_*[a] \not\subset Q_m[a](v), \forall m \in \mathbb{N}.$$

Proof. Let us compute the minimum distance of the set $V_*[a]$ from the origin of the b-plane. By (3.1) we get

$$|b_*(\theta;a)|^2 = \Re[a]^2 + \Im[a]^2 - 2\Im[a]\theta + \theta^2. \qquad (3.4)$$

Since $|b_*(\theta;a)|$ diverges for $|\theta| \longrightarrow \infty$ and

$$\frac{\partial |b_*(\theta;a)|^2}{\partial \theta} = 2(\theta - \Im[a]),$$

we have:

$$\min_{\theta\in\mathbb{R}} |b_*(\theta;a)| = |b_*(\Im[a];a)| = |\Re[a]|. \qquad (3.5)$$

The same computation for the boundary locus generated by the $ETR_2(v,m)$, by (3.3),

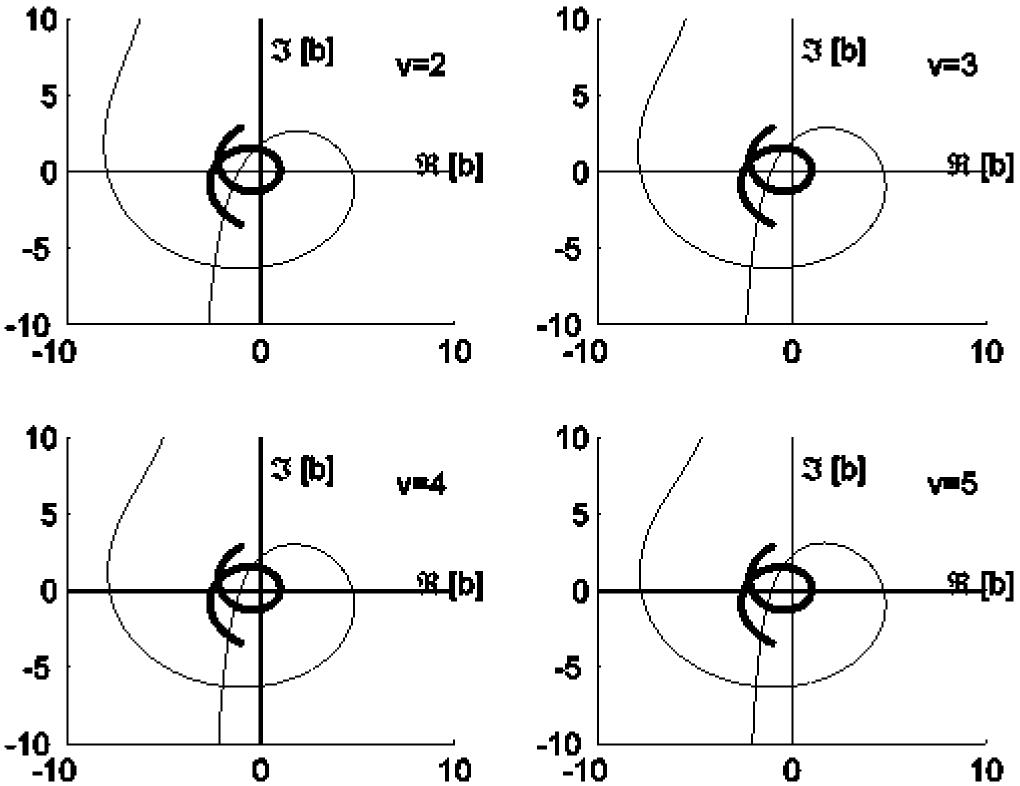

Figure 3.2: The stability region (inside bold line) versus the $\boldsymbol{ETR_2(v, m=1)}$ stability region (inside fine line), corresponding to $\boldsymbol{\Re[a] = -1, \Im[a] = 6}$.

$$|b_*(\theta;a)|^2 = \Re[a]^2 + \Im[a]^2 + 4mq(v,\frac{\theta}{m})\Im[a] + 4m^2q^2(v,\frac{\theta}{m}), \qquad (3.6)$$

and, by some algebraic manipulations and Lemma 2.2(i),

$$\frac{\partial |b_m(\theta;a,v)|^2}{\partial \theta} = 0 \Leftrightarrow \theta = \theta^*, where\ \ 2mq(v,\frac{\theta^*}{m}) + \Im[a] = 0.$$

On this basis we can prove that

$$\min_{\theta\in(-m\pi,m\pi)} |b_m(\theta;a,v)| = |b_m(\theta^*;a,v)| = |\Re[a]|.$$

urthermore we observe that

$$b_*(\theta;a) \neq b_m(\theta;a,v)\ \ if\ \ \Im[a] \neq 0. \qquad (3.8)$$

from Lemma 2.2(ii). However, by (3.5) and (3.7),

$$|b_*(\Im[a];a)| = |b_m(\theta^*;a,v)| = |\Re[a]|. \tag{3.9}$$

Now, by well know results concerning A-stability, we have that the origin of the b-plane belongs to both sets $\Xi_*[a]$ and $Q_m[a](v)$. As a consequence we get

$$b_*(\Im[a];a) \in \partial\Xi_*[a], b_m(\theta^*;a,v) \in \partial Q_m[a](v). \tag{3.10}$$

Finally, by (3.9)-(3.10), we are in a position to state the following relations:

$\Xi_*[a] \not\subset Q_m[a](v)$,

$Q_m[a](v) \not\subset \Xi_*[a]$,

for any m and any a considered in the hypotheses (see Fig.3.2).

The central result is an immediate consequence of Lemma 3.2.

4. NUMERICAL EXPERIMENTS

In order to illustrate the Theorem 3.3. Consider the complex coefficient test equation

$$\begin{cases} y'(t) = ay(t) + by(t-1), t > 0, \\ y(t) = g(t) = e^{at}sin(\frac{\pi}{2}t), t \leq 0. \end{cases} \tag{4.1}$$

where $a = -1 + 6i, b = -1.5 - 0.8i$ and $i = \sqrt{-1}$. The $ETR_2(v,m)$ (2.4) with $h = 1/m(m > 0)$ is applied to equation (4.1) for studying the numerical stability (τ-stable). It shows that the exact solution of (4.1) is asymptotically stable.

For the $ETR_2(v,m)$, equation (2.4) requires the additional initial conditions which can be gotten by $y_i = y(ih) = g(t_i) = e^{at_i}sin(\frac{\pi}{2}t_i)$, for $i \leq 0$ and by additional initial equations obtained by additional methods, and $v-1$ additional final conditions, which can be gotten by additional final equations obtained by additional methods [13]. It can be shown that if the additional methods are appropriately chosen, the stability properties of the global methods are inherited by the main formula (2.4) [13].

As an example, we mention the fourth-order ETR$_2$ ($v = 2$)

$$\frac{1}{12}(y_{n+1} + 9y_n - 9y_{n-1} - y_{n-2}) = \frac{h}{2}(f_n + f_{n-1}), n = 2, \dots, N-1,$$

which can be completed with the following additional equations:

$$\frac{1}{24}(-y_3 + 9y_2 + 9y_1 - 17y_0) = \frac{h}{4}(3f_1 + f_0),$$

$$\frac{1}{24}(y_{N-3} - 9y_{N-2} - 9y_{N-1} + 17y_N) = \frac{h}{4}(3f_{N-1} + f_N),$$

obtained by additional fourth-order methods.

For the six-order ETR$_2$ ($v = 3$)

$$\frac{1}{120}(-y_{n+2} + 15y_{n+1} + 80y_n - 80y_{n-1} - 15y_{n-2} + y_{n-3})$$
$$= \frac{h}{2}(f_n + f_{n-1}), n = 3, \dots, N-1,$$

which can be completed with the following additional equations, obtained from sixth order methods,

$$\frac{-1}{360}(3y_5 - 25y_4 + 100y_3 - 300y_2 + 25y_1 + 197y_0) = \frac{h}{6}(5f_1 + f_0),$$

$$\frac{-1}{180}(-y_5+10y_4-60y_3-80y_2+125y_1+6y_0)=\frac{h}{3}(2f_2+f_1),$$
$$\frac{1}{180}(-y_{N-4}+10y_{N-3}-60y_{N-2}-80y_{N-1}+125y_N+6y_{N+1})=\frac{h}{3}(2f_{N-1}+f_N),$$
$$\frac{1}{360}(3y_{N-4}-25y_{N-3}+100y_{N-2}-300y_{N-1}+25y_N+197y_{N+1})=\frac{h}{6}(5f_N+f_{N+1}).$$

For each ETR_2 previously considered, there exists an appropriate choice of the additional methods [13]. The numerical solution (2.4) for the complex equation (4.1) is given in Fig 4.1, Fig 4.2 and Fig 4.3 for the different $ETR_2(v,m)$.

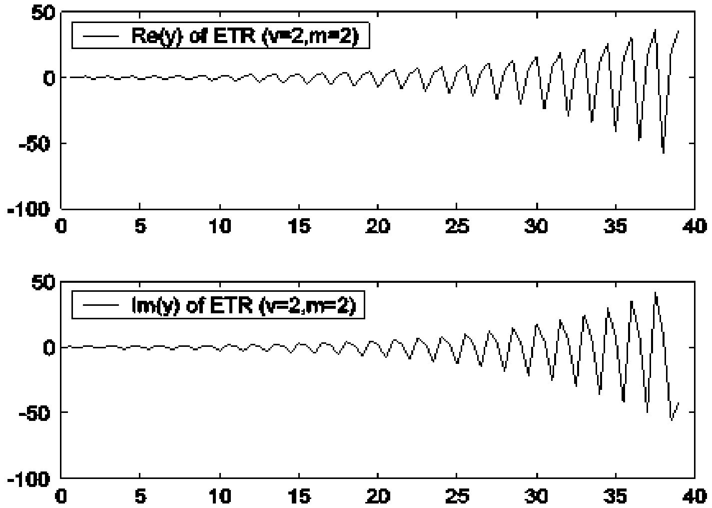

Figure 4.1 Numerical solution of ETR_2s (2.4) for (4.1) with $a=-1+6i, b=-1.5-0.8i$.

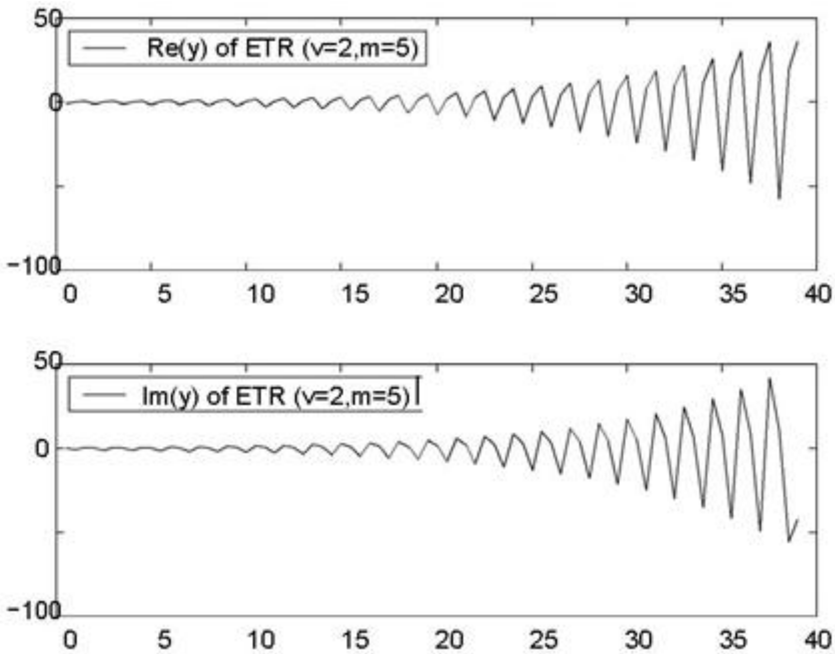

Figure 4.2 Numerical solution of ETR_2s (2.4) for (4.1) with $a=-1+6i, b=-1.5-0.8i$.

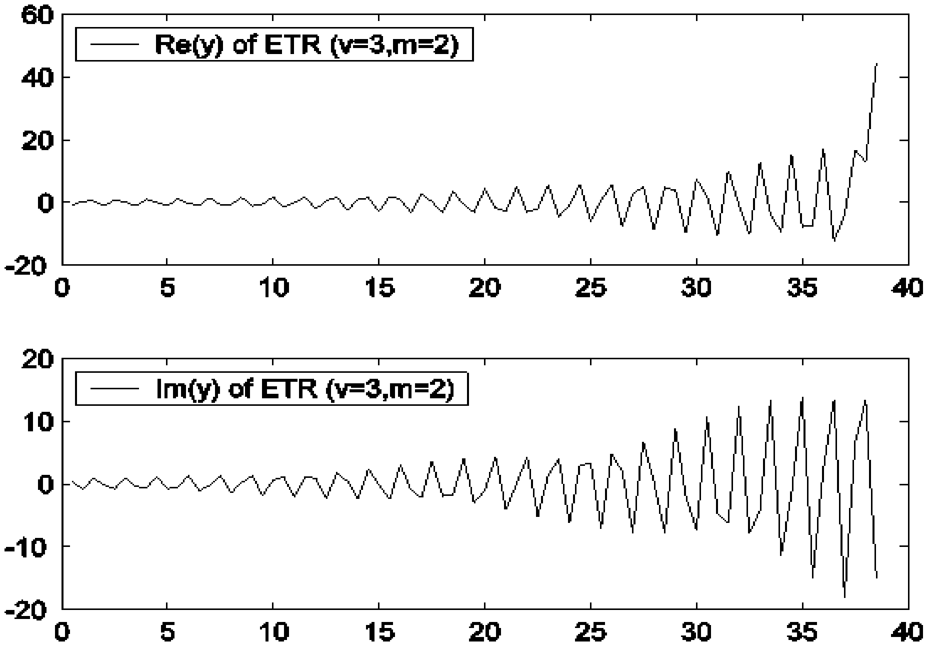

Figure 4.3:Numerical solution of ETR_2s (2.4) for (4.1) with $a = -1 + 6i, b = -1.5 - 0.8i$.

Fig 4.1 and Fig4.2 give numerical solution of ETR_2s (different stepsize m) for (4.1). However, the numerical solutions of Fig 4.1 and Fig 4.2 are not stable for the different ETR_2s$(2, m)$. Fig 4.1 and Fig 4.3 give numerical solution of ETR_2s (different order v) for (4.1). Fig 4.1-4.3 demonstrate that the ETR_2s cannot preserve the asymptotic stability of its exact solution when $a \in \mathbb{C}$. So, the ETR_2s are no τ-stable. These numerical examples confirm our theoretical findings.

5. Conclusions

The delay dependent stable regions of the extended trapezoidal rules of second kind ($ETR_2(v, m), v < 6$), which are a class of BVMs, are displayed for the test equation of DDEs. Furthermore, It is showed ETR_2s for $v < 6$ cannot preserve the delay-dependent stability of the complex coefficient test equation considered. Before concluding the section 3 and section 4, we point out that the $ETR_2(v, m)$ for $v \geq 6$ has not yet been studied. However, $ETR_2(v, m)$ are a class of BVMs and constructed using the same law. If the coefficients of the $ETR_2(v, m)$ for $v \geq 6$ are given, we conjecture that the ETR_2s(v, m) for $v \geq 6$ has the same result as the $ETR_2(v, m)$ for $v < 6$ in this paper, which $ETR_2(v, m)$ for $v \geq 6$ can not preserve the delay-dependent stability of the complex coefficient test equation considered. This will be one of the next steps we need to do.

References

[1] A. Bellen and M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford University Press, Oxford, 2003.

[2] N. Guglielmi, On asymptotic stability properties for Runge-Kutta methods for delay differential equations, Numer. Math. 77(1997), 467-485.

[3] N. Guglielmi, Delay dependent stability region of θ-methods for delay differential equations, IMA J. Numer. Anal. 18(1998), 339-418.

[4] S.F.Wu and S.G.Gan, Analytical and numerical stability of neutral selay-integro -differential equations and neutral delay partial differential equations, Comput. Math. Appl. 55 2008, 2426-2443.

[5] J Zhao , Y Xu , X Li, Y Fan:Delay-dependent stability of symmetric boundary value methods for second order delay differential equations with three parameters. Numerical Algorithms , 2015 , 69 (2) :321-336.

[6] J Ma , S Xiang, A Collocation Boundary Value Method for Linear Volterra Integral Equations .Journal of Scientific Computing , 2017 , 71 (1) :1-20 .

[7] L Aceto, P Ghelardoni, C Magherini, PGSCM: A family of P-stable Boundary Value Methods for second-order initial value problems. Elsevier Science Publishers B. V. , 2012 , 236 (16) :3857-3868.

[8] L Aceto, P Ghelardoni , C Magherini , P-stable boundary value methods for second order IVPs .Icnaam: International Conference of Numerical A... , 2012 , 1479 (1) :1165-1168 .

[9] Abdelhameed Nagy Abdo, A.M., Numerical solution of stiff and singularly perturbed problems for ordinary differential and Volterra-type equations. Phd Thesis, Università di Bari (2012).

[10] P. Amodio and F. Mazzia, A boundary value approach to the numerical solution of ODEs by multistep methods, Jour. of Diference Eq. and Appl. 1(1995), 353-367.

[11] L Brugnano, F Iavernaro , D Trigiante, Reprint of Analysis of Hamiltonian Boundary Value Methods (HBVMs): A class of energy-preserving Runge–Kutta methods for the numerical solution of polynomial Hamiltonian systems.Communications in Nonlinear Science and Numerical Simulation.2015 , 20 (3) :650-667.

[12] L.Brugnano and D. Triglante, Boundary value methods: The third way between linear mutistep and runge-kutta methods, Computer Math. Applic. 36(1998), 269-284.

[13] L.Brugnano and D. Triglante, Solving Differetial problem by multistep initial and boundary value method s, Gordan and Breach, Amsterdam , 1998.

[14] L.Brugnano, Essentially symplectic boundary value methods for linear Hamiltonian systems, J. Comput. Math. 15(1997), 233-252.

[15] E.Hairer and G.Wanner, Solving ordinary differential equations I, Springer, Berlin, 1991.

[16] C. T. H. Baker and C. A. H. Paul, Computing stability regions-Runge-Kutta methods for delay differential equations, IMA J. Numer. Anal. 14(1994), 347-362.

[17] O. Diekmann, S. A. Van Gils, S. M. Verduin Lunel and H. -O. Walther, Delay equations: Functional-,Complex-, and Nonlinear Analysis, Spinger-Verlag, Berlin, 1995.