ADAPTIVE CONTROL AND SYNCHRONIZATION OF A GENERALIZED LOTKA-VOLTERRA SYSTEM

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ABSTRACT

The Lotka-Volterra equations are a system of equations proposed to provide a simplified model of two-species predator-prey population dynamics. In this paper, we investigate the problem of adaptive chaos control and synchronization of a generalized Lotka-Volterra system discovered by Samardzija and Greller (1988). The Samardzija-Greller model is a two-predator, one-prey generalization of the Lotka-Volterra system. First, adaptive control laws are designed to stabilize the generalized Lotka-Volterra system to its unstable equilibrium point at the origin based on the adaptive control theory and Lyapunov stability theory. Then adaptive control laws are derived to achieve global chaos synchronization of identical generalized Lotka-Volterra systems with unknown parameters. Numerical simulations are shown to validate and demonstrate the effectiveness of the proposed adaptive control and synchronization schemes for the generalized Lotka-Volterra system.

KEYWORDS

Adaptive Control, Stabilization, Chaos Synchronization, Generalized Lotka-Volterra Chaotic System.

1. INTRODUCTION

Chaotic systems are nonlinear dynamical systems, which are highly sensitive to initial conditions. The sensitive nature of chaotic systems is usually called as the butterfly effect [1]. In 1963, Lorenz first observed the chaos phenomenon in weather models. Since then, a large number of chaos phenomena and chaos behaviour have been discovered in physical, social, economical, biological and electrical systems.

The control of chaotic systems is to design state feedback control laws that stabilizes the chaotic systems around the unstable equilibrium points. Active control technique is used when the system parameters are known and adaptive control technique is used when the system parameters are unknown [2-4].

Chaos synchronization is a phenomenon that may occur when two or more chaotic oscillators are coupled or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect, which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, synchronizing two chaotic systems is seemingly a very challenging problem in the chaos literature [5-16].
In 1990, Pecora and Carroll [5] introduced a method to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then on, chaos synchronization has been widely explored in a variety of fields including physical systems [6], chemical systems [7], ecological systems [8], secure communications [9-10], etc.

In most of the chaos synchronization approaches, the master-slave or drive-response formalism has been used. If a particular chaotic system is called the master or drive system and another chaotic system is called the slave or response system, then the idea of synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the seminal work by Pecora and Carroll [5], a variety of impressive approaches have been proposed for the synchronization of chaotic systems such as the OGY method [11], active control method [12-16], adaptive control method [17-21], sampled-data feedback synchronization method [22], time-delay feedback method [23], backstepping method [24], sliding mode control method [25-28], etc.

In this paper, we investigate the adaptive control and synchronization of an uncertain generalized Lotka-Volterra system discovered by Samardzija and Greller (1988).

The Samardzija-Greller system is a two-predator, one-prey generalization of the Lotka-Volterra equations. This system has the following interesting properties. First of all, it exhibits chaotic behaviour under more lenient conditions for a generalization of Lotka-Volterra systems. Second, the chaotic behaviour is an example of the “explosive” route to chaos. Finally, in various regions of the phase space, the Samardzija-Greller system evolves on a “fractal torus”.

First, we devise adaptive stabilization scheme using state feedback control for the generalized Lotka-Volterra system about its unstable equilibrium at the origin. Then, we devise adaptive synchronization scheme for identical generalized Lotka-Volterra systems with unknown parameters. The stability results derived in this paper are established using Lyapunov stability theory.

This paper is organized as follows. In Section 2, we give a system description of the generalized Lotka-Volterra system (1988). In Section 3, we derive results for the adaptive chaos control of the generalized Lotka-Volterra system with unknown parameters. In Section 4, we derive results for the adaptive synchronization of identical generalized Lotka-Volterra systems with unknown parameters. In Section 5, we summarize the main results obtained in this paper.

2. System Description

The generalized Lotka-Volterra system ([29], 1988) is described by the dynamics

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1 x_2 + cx_1^2 - ax_1^2 x_3 \\
\dot{x}_2 &= -x_2 + x_1 x_2 \\
\dot{x}_3 &= -bx_3 + ax_1^2 x_3
\end{align*}
\]

where \( x_i, (i=1,2,3) \) are the state variables and \( a, b, c \) are constant positive parameters of the system.

The system (1) is chaotic when the parameter value is taken as

\[
a = 2.9851, \quad b = 3, \quad c = 2
\]

The state orbits of the generalized Lotka-Volterra chaotic system (1) are described in Figure 1.
Figure 1. State Orbits of the Generalized Lotka-Volterra Chaotic System

In [29], it has been shown that that chaotic behaviour of the system (1) is an example of the “explosive” route to chaos and is attributed to the non-transversal saddle connection type bifurcation. Also, it is observed that the chaotic solution of the generalized Lotka-Volterra system (1) portrays a fractal torus in the three-dimensional phase space.

When the parameter values are taken as in (2), the system (1) is chaotic and the system linearization matrix at the equilibrium point \( E_0 = (0, 0, 0) \) is given by

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -3
\end{bmatrix}
\]

which has the eigenvalues

\[
\lambda_1 = 1, \quad \lambda_2 = -1 \quad \text{and} \quad \lambda_3 = -3.
\]

Since \( \lambda_1 \) is a positive eigenvalue, it is immediate from Lyapunov stability theory [30] that the system (1) is unstable at the equilibrium point \( E_0 = (0, 0, 0) \).
3. ADAPTIVE CONTROL OF THE GENERALIZED LOTKA-VOLTERRA CHAOTIC SYSTEM

3.1 Theoretical Results

In this section, we design adaptive control law for globally stabilizing the generalized Lotka-Volterra chaotic system (1) when the parameter value is unknown. Thus, we consider the controlled generalized Lotka-Volterra system described by

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1 x_2 + c x_1^2 - a x_1^2 x_3 + u_1 \\
\dot{x}_2 &= -x_2 + x_1 x_2 + u_2 \\
\dot{x}_3 &= -b x_3 + a x_1^2 x_3 + u_3
\end{align*}
\] (5)

where \(u_1, u_2\), and \(u_3\) are feedback controllers to be designed using the states and estimates of the unknown parameter of the system.

In order to ensure that the controlled system (5) globally converges to the origin asymptotically, we consider the following adaptive control functions

\[
\begin{align*}
\dot{\hat{a}} &= -x_1 + x_1 x_2 - \hat{c} x_1^2 + \hat{a} x_1^2 x_3 - k_1 x_1 \\
\dot{\hat{b}} &= x_2 - x_1 x_2 - k_2 x_2 \\
\dot{\hat{c}} &= \hat{b} x_3 - \hat{a} x_1^2 x_3 - k_3 x_3
\end{align*}
\] (6)

where \(\hat{a}, \hat{b}, \hat{c}\) are the estimate of the parameters \(a, b, c\), respectively and \(k_i, (i = 1, 2, 3)\) are positive constants.

Substituting the control law (6) into the highly chaotic dynamics (5), we obtain

\[
\begin{align*}
\dot{\hat{a}} &= (c - \hat{c}) x_1^2 - (a - \hat{a}) x_1^2 x_3 - k_1 x_1 \\
\dot{\hat{b}} &= -k_2 x_2 \\
\dot{\hat{c}} &= -(b - \hat{b}) x_3 + (a - \hat{a}) x_1^2 x_3 - k_3 x_3
\end{align*}
\] (7)

Let us now define the parameter estimation error as

\[
\begin{align*}
e_a &= a - \hat{a} \\
e_b &= b - \hat{b} \\
e_c &= c - \hat{c}
\end{align*}
\] (8)

Using (8), the closed-loop dynamics (7) can be written compactly as

\[
\begin{align*}
\dot{x}_1 &= e_a x_1^2 - e_a x_1^2 x_3 - k_1 x_1 \\
\dot{x}_2 &= -k_2 x_2 \\
\dot{x}_3 &= -e_b x_3 + e_a x_1^2 x_3 - k_3 x_3
\end{align*}
\] (9)

For the derivation of the update law for adjusting the parameter estimates, the Lyapunov approach is used.

Consider the quadratic Lyapunov function

\[
V = \frac{1}{2} \left( x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2 \right)
\] (10)
which is a positive definite function on $\mathbb{R}^6$.

Note also that

\[
\begin{align*}
\dot{e}_a &= -\hat{\alpha} \\
\dot{e}_b &= -\hat{\beta} \\
\dot{e}_c &= -\hat{\gamma}
\end{align*}
\]

Differentiating $V$ along the trajectories of (9) and using (11), we obtain

\[
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 + e_a \left[ -x_1^3 x_3 + x_1^2 x_3^2 - \hat{\alpha} \right] + e_b \left[ -x_3^2 - \hat{\beta} \right] + e_c \left[ x_1^3 - \hat{\gamma} \right]
\]

(12)

In view of Eq. (12), the estimated parameters are updated by the following law:

\[
\begin{align*}
\dot{\alpha} &= -x_1^3 x_3 + x_1^2 x_3^2 + k_4 e_a \\
\dot{\beta} &= -x_3^2 + k_5 e_b \\
\dot{\gamma} &= x_1^3 + k_6 e_c
\end{align*}
\]

(13)

where $k_4, k_5, k_6$ are positive constants.

Substituting (13) into (12), we get

\[
\dot{V} = -k_1 x_1^2 - k_2 x_2^2 - k_3 x_3^2 - k_4 e_a^2 - k_5 e_b^2 - k_6 e_c^2
\]

(14)

which is a negative definite function on $\mathbb{R}^6$.

Thus, by Lyapunov stability theory [30], we obtain the following result.

**Theorem 1.**

The generalized Lotka-Volterra chaotic system (5) with unknown parameters is globally and exponentially stabilized for all initial conditions $x(0) \in \mathbb{R}^3$ by the adaptive control law (6), where the update law for the parameter is given by (13) and $k_i$, $(i = 1, \ldots, 6)$ are positive constants.

**2.2 Numerical Results**

For the numerical simulations, the fourth order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the generalized Lotka-Volterra chaotic system (5) with the adaptive control law (6) and the parameter update law (13).

The parameters of the generalized Lotka-Volterra chaotic system (5) are selected as

\[
a = 2.9851, \quad b = 3, \quad c = 2
\]

For the adaptive and update laws, we take $k_i = 4$, $(i = 1, 2, \ldots, 6)$.

Suppose that the initial value of the parameter estimates are taken as

\[
\hat{\alpha}(0) = 5, \quad \hat{\beta}(0) = 8, \quad \hat{\gamma}(0) = 1
\]

The initial values of the generalized Lotka-Volterra chaotic system (5) are taken as

\[
x_1(0) = 16, \quad x_2(0) = 24, \quad x_3(0) = 18
\]
When the adaptive control law (6) and the parameter update law (13) are used, the controlled generalized Lotka-Volterra chaotic system (5) converges to the equilibrium $E_0 = (0, 0, 0)$ exponentially as shown in Figure 2.

Figure 3 shows that the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ converge to the actual values of the system parameters, viz. $a = 2.9851$, $b = 3$ and $c = 2$. 

Figure 2. Time Responses of the Controlled Generalized Lotka-Volterra System

Figure 3. Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$
4. ADAPTIVE SYNCHRONIZATION OF IDENTICAL GENERALIZED LOTKA-VOLTERRA CHAOTIC SYSTEMS

4.1 Theoretical Results

In this section, we discuss the adaptive synchronization of identical generalized Lotka-Volterra chaotic systems with unknown parameters.

As the master system, we take the generalized Lotka-Volterra dynamics described by

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1x_2 + cx_1^2 - ax_1^2x_3 \\
\dot{x}_2 &= -x_2 + x_1x_2 \\
\dot{x}_3 &= -bx_3 + ax_1^2x_3
\end{align*}
\]

where \( x_i, \ (i = 1, 2, 3) \) are the state variables and \( a, b, c \) are the unknown system parameters.

As the slave system, we take the controlled generalized Lotka-Volterra dynamics described by

\[
\begin{align*}
\dot{y}_1 &= y_1 - y_1y_2 + cy_1^2 - ay_1^2y_3 + u_1 \\
\dot{y}_2 &= -y_2 + y_1y_2 + u_2 \\
\dot{y}_3 &= -by_3 + ay_1^2y_3 + u_3
\end{align*}
\]

where \( y_i, \ (i = 1, 2, 3) \) are the state variables and \( u_i, \ (i = 1, 2, 3) \) are the nonlinear controllers to be designed.

The synchronization error is defined by

\[
\epsilon_i = y_i - x_i, \quad (i = 1, 2, 3)
\]

Then the error dynamics is obtained as

\[
\begin{align*}
\dot{\epsilon}_1 &= \epsilon_1 - y_1y_2 + x_1x_2 + c(y_1^2 - x_1^2) - a(y_1^2y_3 - x_1^2x_3) + u_1 \\
\dot{\epsilon}_2 &= -\epsilon_2 + y_1y_2 - x_1x_2 + u_2 \\
\dot{\epsilon}_3 &= -b\epsilon_3 + a(y_1^2y_3 - x_1^2x_3) + u_3
\end{align*}
\]

Let us now define the adaptive control functions \( u_1(t), u_2(t), u_3(t) \) as

\[
\begin{align*}
u_1 &= -\epsilon_1 + y_1y_2 - x_1x_2 - \hat{c}(y_1^2 - x_1^2) + \hat{a}(y_1^2y_3 - x_1^2x_3) - k_1\epsilon_1 \\
u_2 &= \epsilon_2 + y_1y_2 + x_1x_2 - k_2\epsilon_2 \\
u_3 &= -b\epsilon_3 - \hat{a}(y_1^2y_3 - x_1^2x_3) - k_3\epsilon_3
\end{align*}
\]

where \( \hat{a}, \hat{b}, \hat{c} \) are the estimates of the parameters \( a, b, c \), respectively, and \( k_1, \ k_2, \ k_3 \) are positive constants.

Substituting the control law (19) into (18), we obtain the error dynamics as

\[
\begin{align*}
\dot{\epsilon}_1 &= (c - \hat{c})(y_1^2 - x_1^2) - (a - \hat{a})(y_1^2y_3 - x_1^2x_3) - k_1\epsilon_1 \\
\dot{\epsilon}_2 &= -k_2\epsilon_2 \\
\dot{\epsilon}_3 &= -(b - \hat{b})\epsilon_3 + (a - \hat{a})(y_1^2y_3 - x_1^2x_3) - k_3\epsilon_3
\end{align*}
\]

Let us now define the parameter estimation error as
Substituting (21) into (20), the error dynamics simplifies to


de_1 = e_a(y_1^2 - x_1^2) - e_a(y_3^2 y_3 - x_1^2 x_3) - k_1 e_1
\hat{e}_2 = -k_3 e_2
\hat{e}_3 = -e_y e_3 + e_a(y_1^2 y_3 - x_1^2 x_3) - k_3 e_3

(22)

For the derivation of the update law for adjusting the estimate of the parameter, the Lyapunov approach is used.

Consider the quadratic Lyapunov function

\[ V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2) \]

(23)

which is a positive definite function on \( \mathbb{R}^6 \).

Note also that

\[ \hat{e}_a = -\dot{a} \]
\[ \hat{e}_b = -\dot{b} \]
\[ \hat{e}_c = -\dot{c} \]

(24)

Differentiating \( V \) along the trajectories of (22) and using (24), we obtain

\[ \dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a[(e_3 - e_1)(y_1^2 y_3 - x_1^2 x_3) - \hat{a}] + e_b[-e_2^2 - \dot{b}] + e_c[e_1(y_1^2 - x_1^2) - \dot{c}] \]

(25)

In view of Eq. (25), the estimated parameter is updated by the following law:

\[ \dot{\hat{a}} = (e_3 - e_1)(y_1^2 y_3 - x_1^2 x_3) + k_4 e_a \]
\[ \dot{\hat{b}} = -e_2^2 + k_5 e_b \]
\[ \dot{\hat{c}} = e_1(y_1^2 - x_1^2) + k_6 e_c \]

(26)

where \( k_4, k_5, k_6 \) are positive constants.

Substituting (24) into (23), we get

\[ \dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_a^2 - k_5 e_b^2 - k_6 e_c^2 \]

(27)

which is a negative definite function on \( \mathbb{R}^6 \).

Thus, by Lyapunov stability theory [30], it is immediate that the synchronization error and the parameter error decay to zero exponentially with time for all initial conditions.

Hence, we have proved the following result.

**Theorem 2.**

The identical generalized Lotka-Volterra systems (15) and (16) with unknown parameters are globally and exponentially synchronized for all initial conditions by the adaptive control law
(19), where the update law for parameter is given by (26) and $k_i, (i = 1, \ldots, 6)$ are positive constants.

### 3.2 Numerical Results

For the numerical simulations, the fourth order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the two systems of differential equations (15) and (16) with the adaptive control law (19) and the parameter update law (26).

Here, we take the system parameter values as

\[ a = 2.9851, \quad b = 3, \quad c = 2 \]

and the gains as $k_i = 4$ for $i = 1, 2, 3, 4, 5, 6$.

We take the initial value of the parameter estimates as

\[ \hat{a}(0) = 3, \quad \hat{b}(0) = 7, \quad \hat{c}(0) = 1. \]

We take the initial state of the master system (15) as $x(0) = (2, 5, 4)$ and the slave system (16) as $y(0) = (1, 6, 8)$.

Figure 4 shows the adaptive chaos synchronization of the identical generalized Lotka-Volterra systems. Figure 5 shows that the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ converge to the actual values of the system parameters $a, b, c$.

![Figure 4. Adaptive Synchronization of the Generalized Lotka-Volterra Systems](image)
5. CONCLUSIONS

In this paper, we applied adaptive control theory for the stabilization and synchronization of the generalized Lotka-Volterra chaotic system (Samardzija and Greller, 1988) with unknown system parameters. First, we designed adaptive control laws to stabilize the generalized Lotka-Volterra chaotic system to its equilibrium point at the origin based on the adaptive control theory and Lyapunov stability theory. Then we derived adaptive synchronization scheme and update law for the estimation of system parameters for identical generalized Lotka-Volterra chaotic systems with unknown parameters. Our synchronization schemes were established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the proposed adaptive control method is very effective and convenient to achieve chaos control and synchronization of the generalized Lotka-Volterra chaotic system. Numerical simulations are presented to demonstrate the effectiveness of the adaptive stabilization and synchronization schemes derived in this paper.

REFERENCES


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