SLIDING MODE CONTROLLER DESIGN FOR HYBRID SYNCHRONIZATION OF HYPERCHAOTIC CHEN SYSTEMS

Sundarapandian Vaidyanathan¹ and Sivaperumal Sampath²

¹Research and Development Centre, Vel Tech Dr. RR & Dr. SR Technical University
Avadi, Chennai-600 062, Tamil Nadu, INDIA
sundarvtu@gmail.com

²Institute of Technology, CMJ University
Shillong, Meghalaya-793 003, INDIA
sivaperumals@gmail.com

ABSTRACT

This paper derives new results for the design of sliding mode controller for the hybrid synchronization of identical hyperchaotic Chen systems (Jia, Dai and Hui, 2010). The synchronizer results derived in this paper for the hybrid synchronization of identical hyperchaotic Chen systems are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the sliding mode control method is very effective and convenient to achieve hybrid synchronization of the identical hyperchaotic Chen systems. Numerical simulations are shown to illustrate and validate the hybrid synchronization schemes derived in this paper for the identical hyperchaotic Chen systems.

KEYWORDS

Sliding Mode Control, Hyperchaos, Hybrid Synchronization, Hyperchaotic Systems, Hyperchaotic Chen System.

1. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the butterfly effect [1]. Chaos is an interesting nonlinear phenomenon and has been extensively studied in the last three decades. A hyperchaotic system is usually characterized as a chaotic system with more than one positive Lyapunov exponent implying that the dynamics expand in more than one direction giving rise to “thicker” and “more complex” chaotic dynamics. The first hyperchaotic system was discovered by Rössler in 1979 [2]. In the last two decades, hyperchaotic systems found many applications in areas such as secure communications, data encryptions, etc.

In most of the chaos synchronization approaches, the master-slave or drive-response formalism is used. If a particular chaotic system is called the master or drive system and another chaotic system is called the slave or response system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

Since the pioneering work by Pecora and Carroll ([3], 1990), chaos synchronization problem has been studied extensively and intensively in the literature [3-35]. Chaos theory has been applied to a variety of fields such as physical systems [4], chemical systems [5], ecological systems [6], secure communications [7-9], etc.
In the last two decades, various schemes have been successfully applied for chaos synchronization such as PC method [3], OGY method [10], active control method [11-18], adaptive control method [19-27], time-delay feedback method [28-29], backstepping design method [30], sampled-data feedback method [31], etc.

So far, many types of synchronization phenomenon have been presented such as complete synchronization [3], generalized synchronization [32], anti-synchronization [33-34], projective synchronization [35], generalized projective synchronization [36-37], etc. Complete synchronization is characterized by the equality of state variables evolving in time, while anti-synchronization (AS) is characterized by the disappearance of the sum of relevant state variables evolving in time. Projective synchronization (PS) is characterized by the fact that the master and slave systems could be synchronized up to a scaling factor.

In generalized projective synchronization (GPS), the responses of the synchronized dynamical states synchronize up to a constant scaling matrix. It is easy to see that the complete synchronization and anti-synchronization are special cases of the generalized projective synchronization where the scaling matrix $\alpha = I$, and $\alpha = -I$, respectively.

In hybrid synchronization of two chaotic systems [38-40], one part of the systems is completely synchronized and the other part is anti-synchronized so that the complete synchronization (CS) and anti-synchronization (AS) co-exist in the systems.

In this paper, we derive new results based on the sliding mode control (SMC) [41-43] for the global chaos synchronization of identical hyperchaotic Chen systems ([44], Jia,Dai and Hui, 2010).

Our new stability results for the hybrid synchronization schemes using sliding mode control (SMC) are established using Lyapunov stability theory [45]. In robust control systems, sliding mode control is often adopted due to its inherent advantages of easy realization, fast response and good transient performance as well as its insensitivity to parameter uncertainties and external disturbances. Sliding mode control have many important applications in Control Systems, Electronics and Communication Engineering.

This paper has been organized as follows. In Section 2, we describe the problem statement and our methodology using sliding mode control. In Section 3, we discuss the hybrid synchronization of identical hyperchaotic Chen systems ([44], 2010). In Section 4, we summarize the main results obtained in this paper.

2. PROBLEM STATEMENT AND OUR METHODOLOGY USING SMC

In this section, we describe the problem statement for the hybrid synchronization for identical chaotic systems and our methodology using sliding mode control (SMC).

Consider the chaotic system described by

$$\dot{x} = Ax + f(x)$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ is the state of the system, $A$ is the $n \times n$ matrix of the system parameters and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear part of the system.

We consider the system (1) as the master or drive system.
As the slave or response system, we consider the following chaotic system described by the dynamics

\[ \dot{y} = Ay + f(y) + u \]  

(2)

where \( y \in \mathbb{R}^n \) is the state of the system and \( u \in \mathbb{R}^m \) is the controller to be designed.

We define the hybrid synchronization error as

\[ e_i = \begin{cases} y_i - x_i, & \text{if } i \text{ is odd} \\ y_i + x_i, & \text{if } i \text{ is even} \end{cases} \]  

(3)

then the error dynamics is obtained as

\[ \dot{e} = Ae + \eta(x, y) + u, \]  

(4)

The objective of the global chaos synchronization problem is to find a controller \( u \) such that

\[ \lim_{t \to \infty} \|e(t)\| = 0 \quad \text{for all } e(0) \in \mathbb{R}^n. \]  

(5)

To solve this problem, we first define the control \( u \) as

\[ u = -\eta(x, y) + Bv \]  

(6)

where \( B \) is a constant gain vector selected such that \( (A, B) \) is controllable.

Substituting (5) into (4), the error dynamics simplifies to

\[ \dot{e} = Ae + Bv \]  

(7)

which is a linear time-invariant control system with single input \( v \).

Thus, the original hybrid synchronization problem can be replaced by an equivalent problem of stabilizing the zero solution \( e = 0 \) of the system (7) by a suitable choice of the sliding mode control.

In the sliding mode control, we define the variable

\[ s(e) = Ce = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \]  

(8)

where \( C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \) is a constant row vector to be determined.

In the sliding mode control, we constrain the motion of the system (7) to the sliding manifold defined by

\[ S = \{ x \in \mathbb{R}^n \mid s(e) = 0 \} \]

which is required to be invariant under the flow of the error dynamics (7).

When in sliding manifold \( S \), the system (7) satisfies the following conditions:

\[ s(e) = 0 \]  

(9)

which is the defining equation for the manifold \( S \) and
which is the necessary condition for the state trajectory $e(t)$ of (7) to stay on the sliding manifold $S$.

Using (7) and (8), the equation (10) can be rewritten as

$$\dot{s}(e) = C [Ae + Bv] = 0$$  \hspace{1cm} (11)

Solving (11) for $v$, we obtain the equivalent control law

$$v_{eq}(t) = -(CB)^{-1}CA e(t)$$ \hspace{1cm} (12)

where $C$ is chosen such that $CB \neq 0$.

Substituting (12) into the error dynamics (7), we obtain the closed-loop dynamics as

$$\dot{e} = \left[ I - B(CB)^{-1}C \right] A e$$ \hspace{1cm} (13)

The row vector $C$ is selected such that the system matrix of the controlled dynamics $\left[ I - B(CB)^{-1}C \right] A$ is Hurwitz, i.e. it has all eigenvalues with negative real parts. Then the controlled system (13) is globally asymptotically stable.

To design the sliding mode controller for (7), we apply the constant plus proportional rate reaching law

$$\dot{s} = -q \text{sgn}(s) - k \ s$$ \hspace{1cm} (14)

where $\text{sgn}(\cdot)$ denotes the sign function and the gains $q > 0$, $k > 0$ are determined such that the sliding condition is satisfied and sliding motion will occur.

From equations (11) and (14), we can obtain the control $v(t)$ as

$$v(t) = -(CB)^{-1}\left[ C(kI + A)e + q \text{sgn}(s) \right]$$ \hspace{1cm} (15)

which yields

$$v(t) = \begin{cases} 
-(CB)^{-1}\left[ C(kI + A)e + q \right], & \text{if } s(e) > 0 \\
-(CB)^{-1}\left[ C(kI + A)e - q \right], & \text{if } s(e) < 0 
\end{cases}$$ \hspace{1cm} (16)

**Theorem 1.** The master system (1) and the slave system (2) are globally and asymptotically hybrid synchronized for all initial conditions $x(0), y(0) \in \mathbb{R}^n$ by the feedback control law

$$u(t) = -\eta(x, y) + Bv(t)$$ \hspace{1cm} (17)

where $v(t)$ is defined by (15) and $B$ is a column vector such that $(A, B)$ is controllable. Also, the sliding mode gains $k, q$ are positive.

**Proof.** First, we note that substituting (17) and (15) into the error dynamics (4), we obtain the
closed-loop error dynamics as
\[
\dot{e} = A e - B (C B)^{-1} \left[ C (k I + A) e + q \text{sgn}(s) \right]
\] (18)

To prove that the error dynamics (18) is globally asymptotically stable, we consider the candidate Lyapunov function defined by the equation
\[
V(e) = \frac{1}{2} s^2(e)
\] (19)

Note that

\[ V(e) \geq 0 \text{ for all } e \in \mathbb{R}^n \text{ and } V(0) = 0 \iff e = 0 \]

Thus, it follows that \( V \) is a positive definite function on \( \mathbb{R}^n \).

Differentiating \( V \) along the trajectories of (18) or the equivalent dynamics (14), we get
\[
\dot{V}(e) = s(e) \dot{s}(e) = -k s^2 - q \text{sgn}(s)s
\] (20)

which is a negative definite function on \( \mathbb{R}^n \).

This calculation shows that \( V \) is a globally defined, positive definite, Lyapunov function for the error dynamics (18), which has a globally defined, negative definite time derivative \( \dot{V} \).

Thus, by Lyapunov stability theory [37], it is immediate that the error dynamics (18) is globally asymptotically stable for all initial conditions \( e(0) \in \mathbb{R}^n \).

This means that for all initial conditions \( e(0) \in \mathbb{R}^n \), we have
\[
\lim_{t \to \infty} \| e(t) \| = 0
\]

Hence, it follows that the master system (1) and the slave system (2) are globally and asymptotically synchronized for all initial conditions \( x(0), y(0) \in \mathbb{R}^n \).

This completes the proof. ■

2. SLIDING CONTROLLER DESIGN FOR HYBRID SYNCHRONIZATION OF IDENTICAL HYPERCHAOTIC CHEN SYSTEMS

3.1 Theoretical Results
In this section, we apply the sliding mode control results of Section 2 to derive state feedback control laws for the hybrid synchronization of identical hyperchaotic Chen systems ([44], Jia, Dai and Hui, 2010).

Thus, the master system is described by the 4-D Chen dynamics
\[ \begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= 4x_1 - 10x_1x_3 + cx_2 + 4x_4 \\
\dot{x}_3 &= -bx_3 + x_2^2 \\
\dot{x}_4 &= -dx_4
\end{align*} \]

(21)

where \( x_1, x_2, x_3, x_4 \) are state variables and \( a, b, c, d \) are positive, constant parameters of the system.

The slave system is also described by the controlled 4-D Chen dynamics

\[ \begin{align*}
\dot{y}_1 &= a(y_2 - y_1) + u_1 \\
\dot{y}_2 &= 4y_1 - 10y_1y_3 + cy_2 + 4y_4 + u_2 \\
\dot{y}_3 &= -by_3 + y_2^2 + u_3 \\
\dot{y}_4 &= -dy_4 + u_4
\end{align*} \]

(22)

where \( y_1, y_2, y_3, y_4 \) are state variables and \( u_1, u_2, u_3, u_4 \) are the active controllers to be designed using sliding mode control (SMC).

The 4-D systems (21) and (22) are hyperchaotic when

\[ a = 35, \quad b = 3, \quad c = 21 \quad \text{and} \quad d = 2 \]

Figure 1 illustrates the phase portrait of the hyperchaotic Chen system (21).

From Figure 1, it is clear that the strange attractor of the system (21) undergoes hyperchaotic behaviour.

The hybrid synchronization error is defined by

\[ \begin{align*}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 + x_2 \\
e_3 &= y_3 - x_3 \\
e_4 &= y_4 + x_4
\end{align*} \]

(23)
The error dynamics is easily obtained as

\[
\begin{align*}
\dot{e}_1 &= a(e_2 - e_1) - 2ae_2 + u_1 \\
\dot{e}_2 &= 4e_1 + ce_2 + 4e_4 + 8x_1 - 10(y_1 y_3 + x_1 x_3) + u_2 \\
\dot{e}_3 &= -be_3 + y_2^2 - x_2^2 + u_3 \\
\dot{e}_4 &= -de_1 - 2dx_1 + u_4
\end{align*}
\]  

(24)

We write the error dynamics (24) in the matrix notation as

\[
\dot{\eta} = Ae + \eta(x, y) + u
\]

(25)

where

\[
A = \begin{bmatrix}
-a & a & 0 & 0 \\
4 & c & 0 & 4 \\
0 & 0 & -b & 0 \\
-d & 0 & 0 & 0
\end{bmatrix}, \quad \eta(x, y) = \begin{bmatrix}
-2ax_2 \\
8x_1 - 10(y_1 y_3 + x_1 x_3) \\
y_2^2 - x_2^2 \\
-2dx_1
\end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
\]

(26)

The sliding mode controller design is carried out as detailed in Section 2.
First, we set \( u \) as
where $B$ is chosen such that $(A, B)$ is controllable. We take $B$ as

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$  \hspace{1cm} (28)$$

In the hyperchaotic case, the parameter values are $a = 35$, $b = 3$, $c = 21$ and $d = 2$

The sliding mode variable is selected as

$$s = Ce = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} e = -e_1 - 2e_2 + e_4$$  \hspace{1cm} (29)$$

which makes the sliding mode state equation asymptotically stable.

We choose the sliding mode gains as $k = 6$ and $q = 0.2$

We note that a large value of $k$ can cause chattering and an appropriate value of $q$ is chosen to speed up the time taken to reach the sliding manifold as well as to reduce the system chattering.

From Eq. (15), we can obtain $v(t)$ as

$$v(t) = 9.5e_1 - 44.5e_2 - e_4 + 0.1 \text{sgn}(s)$$  \hspace{1cm} (30)$$

Thus, the required sliding mode controller is obtained as

$$u = -\eta(x, y) + Bv$$  \hspace{1cm} (31)$$

where $\eta(x, y), B$ and $v(t)$ are defined as in the equations (26), (28) and (30).

By Theorem 1, we obtain the following result.

**Theorem 2.** The identical hyperchaotic Chen systems (21) and (22) are globally and asymptotically hybrid synchronized for all initial conditions with the sliding mode controller $u$ defined by (31). \[ \blacksquare \]

### 3.2 Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic Chen systems (21) and (22) with the sliding mode controller $u$ given by (31) using MATLAB.

In the hyperchaotic case, the parameter values are given by $a = 35, \ b = 3, \ c = 21$ and $d = 2$. 

The sliding mode gains are chosen as

\[ k = 6 \quad \text{and} \quad q = 0.2 \]

The initial values of the master system (21) are taken as

\[ x_1(0) = 20, \quad x_2(0) = -6, \quad x_3(0) = 15, \quad x_4(0) = 26 \]

and the initial values of the slave system (22) are taken as

\[ y_1(0) = -4, \quad y_2(0) = 26, \quad y_3(0) = -11, \quad y_4(0) = 28 \]

Figure 2 illustrates the hybrid synchronization of the identical hyperchaotic Chen systems.

Figure 3 shows the time history of the synchronization error states \( e_1(t), e_2(t), e_3(t), e_4(t) \).
4. CONCLUSIONS

In this paper, we have derived new results using Lyapunov stability theory for the hybrid synchronization of chaotic systems using sliding mode control. As application of our sliding mode controller design, we derived new hybrid synchronization schemes for the identical hyperchaotic Chen systems (2010). Since the Lyapunov exponents are not required for these calculations, the sliding mode control method is very effective and convenient to achieve hybrid synchronization for the identical hyperchaotic Chen systems. Numerical simulations are also shown to illustrate the effectiveness of the hybrid synchronization results derived in this paper via sliding mode control.

REFERENCES


Authors

Dr. V. Sundarapandian obtained his Doctor of Science degree in Electrical and Systems Engineering from Washington University, Saint Louis, USA under the guidance of Late Dr. Christopher I. Byrnes (Dean, School of Engineering and Applied Science) in 1996. He is a Professor in the Research and Development Centre at Vel Tech Dr. RR & Dr. SR Technical University, Chennai, Tamil Nadu, India. He has published over 230 refereed international publications. He has published over 100 papers in National Conferences and over 50 papers in International Conferences. He is the Editor-in-Chief of International Journal of Mathematics and Scientific Computing, International Journal of Instrumentation and Control Systems, International Journal of Control Systems and Computer Modelling and International Journal of Information Technology, Control and Automation, etc. His research interests are Linear and Nonlinear Control Systems, Chaos Theory and Control, Soft Computing, Optimal Control, Process Control, Operations Research, Mathematical Modelling and Scientific Computing. He has delivered several Key Note Lectures on Linear and Nonlinear Control Systems, Chaos Theory and Control, Scientific Computing using MATLAB, SCILAB, etc.

Mr. S. Sivaperumal obtained his M.E. degree in VLSI Design and B.E. degree in Electronics and Communications Engineering from Anna University, Chennai in the years 2007 and 2005 respectively. He is currently Assistant Professor in the ECE Department, Vel Tech Dr. RR & Dr. SR Technical University, Chennai, India. He is pursuing Ph.D. degree in Electronics and Communications Engineering from Institute of Technology, CMJ University, Shillong, Meghalaya, India. He has published many papers in refereed International Journals. He has also published papers on VLSI and Control Systems in National and International Conferences. His current research interests are Robotics, Communications, Control Systems and VLSI Design.