ON THE 1-2-3-EDGE WEIGHTING AND VERTEX COLORING OF COMPLETE GRAPH

Mohammad Reza Farahani

Department of Applied Mathematics, Iran University of Science and Technology (IUST)
Narmak, Tehran, Iran
Mr_Farahani@Mathdep.iust.ac.ir

ABSTRACT

A weighting of the edges of a graph is called vertex-coloring if the labeled degrees of the vertices yield a proper coloring of the graph. In other words, for some $k \in \mathbb{N}$, let $f:E(G) \rightarrow \{1, 2, \ldots, k\}$ be an integer weighting of the edges of a graph $G=(V(G);E(G))$ which have $n$ vertices and implies that a vertex-coloring $S_v = \sum_{u \in V(G)} f(uv)$ for every vertex $v \in V(G)$. In this paper we obtain for $k=3$ a proper 1-2-3-edge weighting and vertex coloring a family of complete graphs.

KEYWORDS

Network Protocols, Edge-labeling; Vertex-coloring; Complete Graph.

1. INTRODUCTION

For a graph $G=(V(G);E(G))$, there exist a function $f:E(G) \rightarrow \{1, 2, \ldots, k\}$ be an edge weighting of the edges of $G$. In other words, for any two arbitrary vertices $u, v \in V(G)$ and edge $uv \in E(G)$ we will have $f(uv) \in \{1, 2, \ldots, k\}$. Also, $S_v = \sum_{u \in V(G)} f(uv)$ is a color for a vertex $v \in V(G)$ such that $S_v \neq S_u$ for any two arbitrary distinct vertices $u, v$ of $G$ (consider $S = \{S_v \mid v \in V(G)\}$) and therefore, a function $s:V(G) \rightarrow S$ is a proper vertex-coloring for $G$.

In 2002, Karonski, Łuczak and Thomason conjectured that such a weighting with $k=3$ is possible for all such graphs (see Conjecture 1 and references [8,10]). For $k=2$ is not sufficient as seen for instance in complete graphs and cycles of length not divisible by 4. A first constant bound of $k=30$ was proved by Addario-Berry, et.al in 2007 [1], which was later improved to $K=16$ by Addario-Berry's group in [2] and $k=13$ by T. Wang and Q. Yu in 2008, [13]. Recently, its new bounds are $k=5$ and $k=6$ by Kalkowski, et.al [8, 9].

In this note we show that there is a proper 1-2-3-edge weighting and vertex coloring for a family $K_{3q}$ for all integer number $q$ of complete graphs and obtain two above functions $f$ and $s$ for $K_{3q}$ exactly. Thus, we have following theorem that is the main result of this paper.

Theorem 1.1. Consider complete graph $K_{3q}$ with the vertex set $V(K_{3q})=\{v_1, v_2, \ldots, v_{3q}\}$ and the edge set $E(K_{3q})=\{e_{ij}=v_iv_j \mid v_i, v_j \in V(K_{3q})\}$ for every integer number $q$. Thus there are a edge weighting $f:E(K_{3q}) \rightarrow \{1, 2, 3\}$ a vertex coloring $s:V(K_{3q}) \rightarrow S=\{9q-3, 9q-4, \ldots, 7q-2, 7q-3\}$. DOI:10.5121/ijcsa.2013.3302 19
such that the induced vertex weights $S_i = \sum_{j=1, j \neq i}^n f(v_j) \in S$.

2. MAIN RESULTS

At first, before prove Theorem 1.1, we contribute the following definition, which is useful to proving.

Definition 2.1. Let $f$ be a function that obtained from $k$-edge weighting and vertex coloring of a connected graph $G$ as order $n$. By using $f$, we can part the edge set $E(G)$ into $k$ important sets $E(G)_i$, $\forall k \in \mathbb{N}$ and are equal to $E(G)_i = \{v \in E(G) | f(v) = i \}$ $\Rightarrow |E(G)_i| = y_i$. Therefore $m = |E(G)| = \Sigma_y$, $y_i$ and $E(G) = \bigcup_i E(G)_i$.

In particular, for a 1-2-3-edge weighting and vertex coloring of complete graph $K_n$, we have three partitions $E(K_n)_1$, $E(K_n)_2$ and $E(K_n)_3$ as follows:

- $E(K_n)_1 = \{v \in E(K_n) | f(v) = 1 \}$ $\Rightarrow |E(K_n)_1| = y_1$
- $E(K_n)_2 = \{v \in E(K_n) | f(v) = 2 \}$ $\Rightarrow |E(K_n)_2| = y_2$
- $E(K_n)_3 = \{v \in E(K_n) | f(v) = 3 \}$ $\Rightarrow |E(K_n)_3| = y_3$

such that $m = |E(K_n)| = y_1 + y_2 + y_3$ and $E(K_n)_1 \cup E(K_n)_2 \cup E(K_n)_3 = E(K_n)$ (obviously, $m = \binom{2}{2} = \frac{n(n-1)}{2}$).

Proof of Theorem 1.1. Consider the complete graph $K_{3q}$ $\forall q \in \mathbb{N}$, with $3q$ vertices and $\frac{q}{2}(3q-1)$ edges. For obtain all aims in Theorem 1.1, we present an algorithm for weighting all edges of $K_{3q}$ with labels 1, 2 and 3.

2.1. ALGORITHM FOR 1-2-3-EDGE WEIGHTING AND VERTEX COLORING OF $K_{3q}$ ($q \geq 5$):

1- Choose an arbitrary vertex $v \in V(K_{3q})$ and label all inside edges to $v$ with 3. If we name $v$ by $v_1$, then $v \in V(K_{3q})$, $f(v_1u) = 3$ and $S_i = 3(3q-1)$.

2- Choose one of adjacent vertices with $v_1$ (we name $v_2$) and label all inside edges to $v_2$ with 3; except an edge $v_1u$ (that we name $u$ by $v_{3q}$), then for all $u \in V(K_{3q}), u \neq 3q, f(v_2u) = 3$ and $f(v_3v_{3q}) = 2$. Thus $S_2 = S_1 - 1 = 9q - 4$.

3- Choose one of adjacent vertices with $v_1$, $v_2$ and name $v_3$. We label all inside edges to $v_3$ with 3; except two edges $v_1v_{3q}, v_1u$, then $\forall w \in V(K_{3q}), w \neq 3q, f(v_1w) = 3$ and $f(v_1v_{3q}) = 2$ (we name $u$ by $v_{3q}$). Thus $S_3 = S_2 - 1 = 9q - 5$.

4- Choose one of adjacent vertices with $v_1, v_2, v_3$ and name $v_4$. We label all inside edges to $v_4$ with 3; except two edges $v_2v_{3q}, v_3v_{3q-1}$ then $\forall w \in V(K_{3q}), w \neq 3q, f(v_1w) = 3$, $f(v_1v_{3q}) = 2$ and $f(v_3v_{3q-1}) = 1$. Thus $S_4 = S_3 - 1 = 9q - 6$.

I- (2 $\leq q+1$) Choose an arbitrary vertex $v \in V(K_{3q})$ that didn't choose above (we name $v_i$). So if $l$ be even, then we label all edge $v_l$ with 3 for $1 \neq 3q - \frac{1+1}{2}$, all edge $v_l$ with 1 for $3q \leq 3q - \frac{1+1}{2} + 2$ and label $v_{l}v_{h}$ (h = $3q - \frac{1}{2} + 1$) with 2.

Else, if $l$ be odd, then we label all edge $v_l$ with 3 for $1 \neq 3q - \frac{1+1}{2}$, all edge $v_l$ with 1 for $3q \leq 3q - \frac{1+1}{2} + 3$ and label $v_{l}v_{h}$ (h = $3q - \frac{1+1}{2} + 1, 3q - \frac{1+1}{2} + 2$) with 2.

In other words,
\[ \forall i = 2, \ldots, 2q \quad f(v_i, v_{i+1}) = \ldots = f(v_{3q-1}, v_{3q}) = 3, \quad 2f(v_{3q}, v_{3q+1}) = 2, \quad f(v_{3q+1}, v_{3q+2}) = 2f(v_{3q+2}, v_{3q+3}) = \ldots = f(v_{3q+4}, v_{3q+5}) = 1. \]

And obviously, \( S_i = S_{i-1} - 9q - 2i \).

\( 2Q+2 \)- Choose one of adjacent vertices with \( v_1, v_2, \ldots, v_{2q+1} \) and name \( v_{2q+2} \). Label two edges \( v_{3q-1}v_{3q} \) and \( v_{3q+4}v_{3q+5} \) with label 1 respectively and other inside edges to \( v_{3q+2} \) with 3, that haven't a label. Therefore \( S_{2q+2} = S_{2q+1} - 3 = 7q - 2. \)

\( J \)- \((3q-2 \geq 2q + 3) = 3q - (q - 3)) \) Choose a remaining vertex and name \( v_j \). Now, let \( j = 3q - h \) \((q - 3 \geq h \geq 2)\) and label three edges \( v_jv_k \) for \( k = 2h + 1, 2h + 2, 2h + 3 \) with label 2. Also, label all edges \( v_jv_k \) with label 3 for \( 1 \leq k \leq 2h = 6q - 2j \) and other edge \( v_jv_k \) give label 1 \((6q-2j+4 = 2h+4 \leq k \leq 3q)\). Then obviously \( S_j = S_{j-1} - 4 = 7q - 6 - 4j \).

\( 3Q-2 \) For \( v_{3q-2} \), there are \( f(v_{3q-2}v_1) = f(v_{3q-2}v_2) = f(v_{3q-2}v_3) = 3, \quad f(v_{3q-2}v_4) = f(v_{3q-2}v_5) = f(v_{3q-2}v_6) = f(v_{3q-2}v_7) = 2 \). Thus \( S_{3q-2} = S_{3q-3} - 2 = 3q + 10 \).

\( 3Q-1 \) For \( v_{3q-1} \), there are \( f(v_{3q-1}v_1) = f(v_{3q-1}v_2) = 3, \quad f(v_{3q-1}v_3) = f(v_{3q-1}v_4) = f(v_{3q-1}v_5) = 2 \). Thus \( S_{3q-1} = S_{3q-2} - 4 = 3q + 6 \).

\( 3Q- \) Finally, for vertex \( v_{3q} \), the edge \( v_{3q}v_1 \) labeled with 3 and \( v_{3q}v_2 \) and \( v_{3q}v_3 \) labeled with 2. Also all edge \( v_{3q}v_i \) \((i = 4, \ldots, 3q-1)\) labeled with one. Thus \( S_{3q} = S_{3q-1} + 3 = 3q + 3 \).

It is obvious that by running this algorithm on complete graph \( K_{3q} \) for \( q = 5, 6, \ldots \), we can obtain its 1-2-3-edge weighting and vertex coloring. For small number 1,2,3,4, reader can see following figures. The 1-2-3-edge weighting and vertex coloring of \( K_5, K_6, K_9 \) and \( K_{12} \) aren't taken some steps of above algorithm. These graphs are shown in Figure 1 and Figure 2. Also, reader can see the running algorithm on \( K_{15} \) in Figure 3.

Thus, by obtaining two functions

The edge weighting \( f: E(K_{3q}) \rightarrow \{1, 2, 3\} \) and the vertex coloring \( s: V(K_{3q}) \rightarrow S = \{9q - 3, 9q - 4, \ldots, 7q - 2, 7q - 3, 7q - 6, 7q - 10, \ldots, 3q + 10, 3q + 6, 3q + 3\} \), proof of Theorem 1.1 is completed.
3. OPEN PROBLEMS AND CONJECTURES

In this section, we calculate some result that concluded from Theorem 1.1 and the algorithm for 1-2-3-edge weighting and vertex coloring of $K_{3q}$. In continue, we introduce some open problem and conjectures, which some of them maybe could be solved in the near future, by other reader.

Lemma 3.1. Consider the 1-2-3-edge weighting and vertex coloring of $K_{3q}$ that obtained from above algorithm and using denotations of Definition 2.1, we have $\gamma_{3q} = \frac{3q^3 - 5q + 2}{2}$, $\beta_{3q} = 3q - 2$ and $\alpha_{3q} = 3q^2 - 2q + 1$.

Proof. The proving of lemma is easy; by refer to the algorithm for 1-2-3-edge weighting and vertex coloring of $K_{3q}$ in proof of Theorem 1.1. Since obviously, $q$ and $2q$ edges with weight 2 are inside to the vertices $v_2, v_4, \ldots, v_{2q}$ and $v_3, v_6, \ldots, v_{2q+1}$, respectively. Also, 3 edges with weight 2 are inside the vertex $v_i$ ($i=2q+2, \ldots, 3q-1$) and two edges with weight 2 are inside to $v_{3q}$. Thus $\beta_{3q} = \frac{q + 2q + 3(q - 2) + 2}{2} = 3q - 2$. 

Figure 2. Two example of algorithm to attain a 1-2-3-edge weighting and vertex coloring of $K_9$ and $K_{12}$.

Figure 3. An example of algorithm to attain a 1-2-3-edge weighting and vertex coloring of $K_{15}$. 

22
Conjecture 3.2. (The 1-2-3-conjecture [6,8,10]) Every connected graph \(G=(V,E)\) non-isomorphic to \(K_2\) (with at least two edges) has an edge weighting \(f:E \to \{1,2,3\}\) and vertex coloring \(s:V \to \{n-1,\ldots,3n-3\}\).

Conjecture 3.3. (n vertex coloring) There are distinct numbers of \(S_v's, v \in V(G)\) of a graph \(G\) of order \(n\), for a 1-2-3-edge weighting and vertex coloring.

Conjecture 3.4. (The 1,2-conjecture [7,11,12]) Every graph \(G\) has a coloring chip configuration \(c:V \cup E \to \{1,2\}\).

Conjecture 3.5. (Antimagic weighting [3,5]) For every connected graph \(G\) (with at least two edges) there is a bijection \(c:E \to \{1,2,\ldots,|E|\}\) such that no two vertices of \(G\) have the same potential.

Conjecture 3.6. (Proper vertex coloring) For all graph \(G\) of order \(n\), there are \(\chi(G)\) numbers of \(S_v's, v \in V(G)\) with this 1-2-3-edge weighting and vertex coloring. Where \(\chi(G)\) is the number of colors of the vertices on the graph \(G\).

Conjecture 3.7. (Lucky weighting, [3,4]) For every graph \(G\), there is a vertex weighting \(c:V \to \{1,2,\ldots,\chi(G)\}\), whose vertex potential \(q_v = \sum_{u \in N(v)} c(u)\) is a proper coloring of \(G\).

ACKNOWLEDGEMENTS

The authors are thankful to Dr. Mehdi Alaeiyan and Mr Hamid Hosseini of Department of Mathematics, Iran University of Science and Technology (IUST) for their precious support and suggestions.

REFERENCES