ON THE 1-2-3-EDGE WEIGHTING AND VERTEX COLORING OF COMPLETE GRAPH

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ABSTRACT

A WEIGHTING OF THE EDGES OF A GRAPH IS CALLED VERTEX-COLORING IF THE LABELED DEGREES OF THE VERTICES YIELD A PROPER COLORING OF THE GRAPH. IN OTHER WORDS, FOR SOME $k\in\mathbb{N}$, *LET* $F:E(G)\rightarrow\{1,2,\ldots,K\}$ *BE AN INTEGER WEIGHTING OF THE EDGES OF A GRAPH* $G=(V(G);E(G))$ *WHICH HAVE N VERTICES AND IMPLIES THAT A VERTEX-COLORING* $S_v := \sum_{v\in V(G)} f(vu)$ *FOR EVERY VERTEX* $V\in V(G)$*. IN THIS PAPER WE OBTAIN FOR* $K=3$ *A PROPER 1-2-3-EDGE WEIGHTING AND VERTEX COLORING A FAMILY OF COMPLETE GRAPHS.*

KEYWORDS

Network Protocols, Edge-labeling; Vertex-coloring; Complete Graph.

1. INTRODUCTION

For a graph $G=(V(G);E(G))$, there exist a function $f:E(G)\rightarrow\{1,2,\ldots,k\}$ be an edge weighting of the edges of G. In other words, for any two arbitrary vertices $u,v\in V(G)$ and edge $uv\in E(G)$ we will have $f(uv)\in\{1,2,\ldots,k\}$. Also, $S_v := \sum_{u\in V(G)} f(vu)$ is a color for a vertex $v\in V(G)$ such that $S_u\neq S_v$ for any two arbitrary distinct vertices u,v of G (consider $S=\{S_v\}_{v\in V(G)}$) and therefore, a function $s:V(G)\rightarrow S$ is a proper vertex-coloring for G.

In 2002, *Karonski*, *Łuczak* and *Thomason* conjectured that such a weighting with k=3 is possible for all such graphs (see Conjecture 1 and references [8,10]). For k=2 is not sufficient as seen for instance in complete graphs and cycles of length not divisible by 4. A first constant bound of k=30 was proved by *Addario-Berry, et.al* in 2007 [1], which was later improved to K=16 by *Addario-Berry*'s group in [2] and $k=13$ by *T. Wang* and *Q. Yu* in 2008, [13]. Recently, its new bounds are $k=5$ and $k=6$ by *Kalkowski, et.al* [8, 9].

In this note we show that there is a proper *1-2-3-edge weighting and vertex coloring* for a family K_{3q} for all integer number q of complete graphs and obtain two above functions f and s for K_{3q} $\forall q\in\mathbb{N}$, exactly. Thus, we have following theorem that is the main result of this paper.

Theorem 1.1. Consider complete graph K_{3q} with the vertex set $V(K_{3q})=\{v_1,v_2,\ldots,v_{3q}\}$ and the edge set $E(K_{3q})=\{e_{ij}=v_iv_j|v_i,v_j\in V(K_{3q})\}$ for every integer number q. Thus there are a edge weighting $f:E(K_{3q})\rightarrow\{1,2,3\}$ a vertex coloring $s:V(K_{3q})\rightarrow S=\{9q\text{-}3,9q\text{-}4,\ldots,7q\text{-}2,7q\text{-}$

3,7q-6,7q-10,…,3q+10,3q+6,3q+3}, such that the induced vertex weights $S_i:=\sum_{j=1, j\neq i}^{n} f(v_i v_j) \in S$.

2. MAIN RESULTS

At first, before prove Theorem 1.1, we contribute the following definition, which is useful to proving.

Definition 2.1. Let f be a function that obtained from k-edge weighting and vertex coloring of a connected graph G as order n. By using f, we can part the edge set $E(G)$ into k important sets $E(G)_i$, $\forall k \in \mathbb{N}$ and are equal to $E(G)_i=\{v_iv_j \in E(G) | f(v_iv_j)=i\} \Rightarrow |E(G)_i|=\gamma_i$. Therefore $m=|E(G)|=\Sigma_i \gamma_i$ and $E(G)=\bigcup_i E(G)_i$.

In particular, for a 1-2-3-edge weighting and vertex coloring of complete graph K_n, we have three partitions $E(K_n)_1$, $E(K_n)_2$ *and* $E(K_n)_3$ as follows:

$E(K_n)_1=\{v_iv_j \in E(K_n) | f(v_iv_j)=1\} \Rightarrow |E(K_n)_1|=\gamma_n$,

$E(K_n)_2=\{v_iv_j \in E(K_n) | f(v_iv_j)=2\} \Rightarrow |E(K_n)_2|=\beta_n$,

$E(K_n)_3=\{v_iv_j \in E(K_n) | f(v_iv_j)=3\} \Rightarrow |E(K_n)_3|=\alpha_n$.

such that $m_n=|E(K_n)|=\gamma+\beta+\alpha$ and $E(K_n)_1 \cup E(K_n)_2 \cup E(K_n)_3=E(K_n)$ (obviously, $m_n=\binom{n}{2}=\frac{n(n-1)}{2}$).

Proof of Theorem 1.1. Consider the complete graph K_{3q} $\forall q \in \mathbb{N}$, with $3q$ vertices and $\frac{3}{2}q(3q-1)$ edges. For obtain all aims in Theorem 1.1, we present an algorithm for weighting all edges of K_{3q} with labels 1, 2 and 3.

2.1. ALGORITHM FOR 1-2-3-EDGE WEIGHTING AND VERTEX COLORING OF K_{3q} $(q \geq 5)$:

1- Choose an arbitrary vertex $v \in V(K_{3q})$ and label all inside edges to v with 3. If we name v by v_1, then $\forall u \in V(K_{3q})$ $f(v_1u)=3$ and $S_1=3(3q-1)$.

2- Choose one of adjacent vertices with v_1 (we name v_2) and label all inside edges to v_2 with 3; except an edge v_2u (that we name u by v_{3q}), then for all $u \in V(K_{3q})$, $u \neq v_{3q}$ $f(v_2u)=3$ and $f(v_2v_{3q})=2$. Thus $S_2=S_1-1=9q-4$.

3- Choose one of adjacent vertices with v_1,v_2 and name v_3. We label all inside edges to v_3 with 3; except two edges v_3v_{3q},v_3u, then $\forall w \in V(K_{3q})$, $w \neq v_{3q},u$ $f(v_3w)=3$ and $f(v_3v_{3q})=f(v_3u)=2$ (we name u by v_{3q-1}). Thus $S_3=S_2-1=9q-5$.

4- Choose one of adjacent vertices with v_1,v_2,v_3 and name v_4. We label all inside edges to v_4 with 3; except two edges v_4v_{3q},v_4v_{3q-1} then $\forall w \in V(K_{3q})$, $w \neq v_{3q},v_{3q-1}$ $f(v_4w)=3$, $f(v_4v_{3q-1})=2$ and $f(v_4v_{3q})=1$. Thus $S_4=S_3-1=9q-6$.

I- $(2 \leq I \leq 2q+1)$ Choose an arbitrary vertex $v \in V(K_{3q})$ that didn't choose above (we name v_i). So

If I be even, then we label all edge v_iv_j with 3 for $1 \leq j \leq 3q-[\frac{i+1}{2}]$, all edge v_iv_j with 1 for $3q \geq j \geq 3q-[\frac{i+1}{2}]+2$ and label v_iv_h ($h=3q-[\frac{i+1}{2}]+1$) with 2.

Else, I be odd, then we label all edge v_iv_j with 3 for $1 \leq j \leq 3q-1-[\frac{i+1}{2}]$, all edge v_iv_j with 1 for $3q \geq j \geq 3q-[\frac{i+1}{2}]+3$ and label v_iv_h ($h=3q-[\frac{i+1}{2}]+1$, $3q-[\frac{i+1}{2}]+2$) with 2.

In other words,

$\forall i = 2,...,2q1 \quad f(v_i v_1) = ... = f(v_i v_{3q-\lceil i/2 \rceil}) = 3, f(v_i v_{3q+1-\lceil i/2 \rceil}) = 2, f(v_i v_{3q+2-\lceil i/2 \rceil}) = ... = f(v_i v_{3q}) = 1.$

$\forall i = 3,...,2q+1 \quad f(v_i v_1) = ... = f(v_i v_{3q-\lceil i/2 \rceil}) = 3, f(v_i v_{3q-\lceil i/2 \rceil}) = f(v_i v_{3q+1-\lceil i/2 \rceil}) = 2, f(v_i v_{3q+2-\lceil i/2 \rceil}) = ... = f(v_i v_{3q}) = 1.$

And obviously, $S_i= S_{i-1}-1=9q-2-i$.

2Q+2- Choose one of adjacent vertices with $v_1,v_2,\ldots,v_{2q+1}$ and name v_{2q+2}. Label two edges $v_{2q+2}v_{3q-1}$ and $v_{2q+2}v_{3q}$ with label 2, 1 respectively and other inside edges to v_{2q+2} with 3, that haven't a label. Therefore $S_{2q+2}=S_{2q+1}-3=7q-6$.

J- ($3q-2\geq J\geq 2q+3=3q-(q-3)$) Choose a remaining vertex and name v_j. Now, let $j=3q-h$ ($q-3\geq h\geq 2$) and label three edges v_jv_k for $k=2h+1$, $2h+2$, $2h+3$ with label 2. Also, label all edges v_jv_k with label 3 for $1\leq k\leq 2h=6q-2j$ and other edge v_jv_k give label 1 ($6q-2j+4=2h+4\leq k\leq 3q$). Then obviously $S_j=S_{j-1}-4=7q-6-4(j-(2q+2))$ and vertex-color of v_j is $S_j=15q+2-4j$.

3Q-2- For v_{3q-2}, there are $f(v_{3q-2}v_1)=f(v_{3q-2}v_2)=f(v_{3q-2}v_3)=f(v_{3q-2}v_4)=3$, $f(v_{3q-2}v_5)=f(v_{3q-2}v_6)=f(v_{3q-2}v_7)=2$ and $f(v_{3q-2}v_8)=\ldots= f(v_{3q-2}v_{3q})=1$. Thus $S_{3q-2}=S_{3q-3}-4=3q+10$.

3Q-1- For v_{3q-1}, there are $f(v_{3q-1}v_1)=f(v_{3q-1}v_2)=3$, $f(v_{3q-1}v_3)= f(v_{3q-1}v_4)= f(v_{3q-1}v_5)=2$ and $f(v_{3q-1}v_6)=\ldots= f(v_{3q-1}v_{3q})=1$. Thus $S_{3q-1}=S_{3q-2}-4=3q+6$.

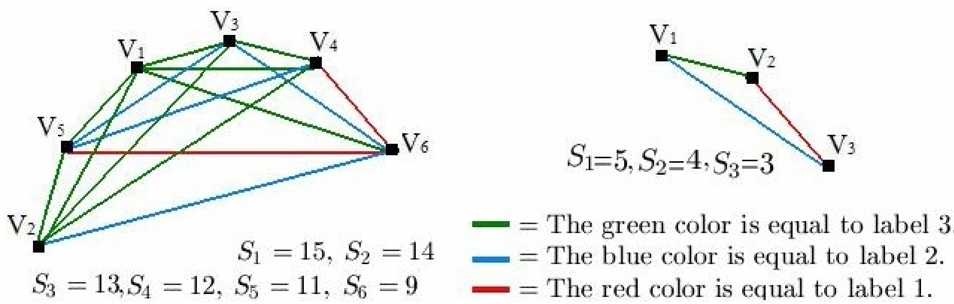

Figure 1. The example of a 1-2-3-edge weighting and vertex coloring of K_3 and K_6.

3Q- Finally, for vertex v_{3q}, the edge $v_{3q}v_1$ labeled with 3 and $v_{3q}v_2$ and $v_{3q}v_3$ labeled with 2. Also all edge $v_{3q}v_i$ $(i=4,\ldots,3q-1)$ labeled with one. Thus $S_{3q}=S_{3q-1}-3=3q+3$.

It is obvious that by running this algorithm on complete graph K_{3q} for $q=5,6,\ldots$, we can obtain its 1-2-3-edge weighting and vertex coloring. For small number 1,2,3,4, reader can see following figures. The 1-2-3-edge weighting and vertex coloring of K_3, K_6, K_9 and K_{12} aren't taken some steps of above algorithm. These graphs are shown in Figure 1 and Figure 2. Also, reader can see the running algorithm on K_{15} in Figure 3. Thus, by obtaining two functions

The *edge weighting* $f:E(K_{3q})\rightarrow \{1,2,3\}$ and the *vertex coloring* $s:V(K_{3q})\rightarrow S=\{9q-3,9q-4,\ldots,7q-2,7q-3,7q-6,7q-10,\ldots,3q+10,3q+6,3q+3\}$, proof of Theorem 1.1 is completed. □

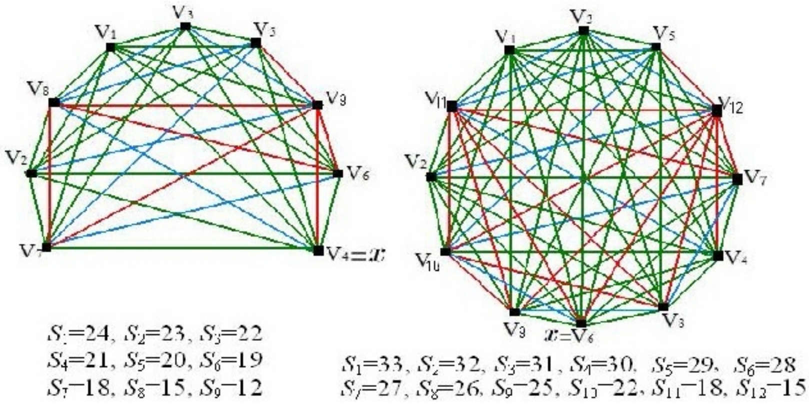

Figure 2. Two example of algorithm to attain a 1-2-3-edge weighting and vertex coloring of K_9 and K_{12}.

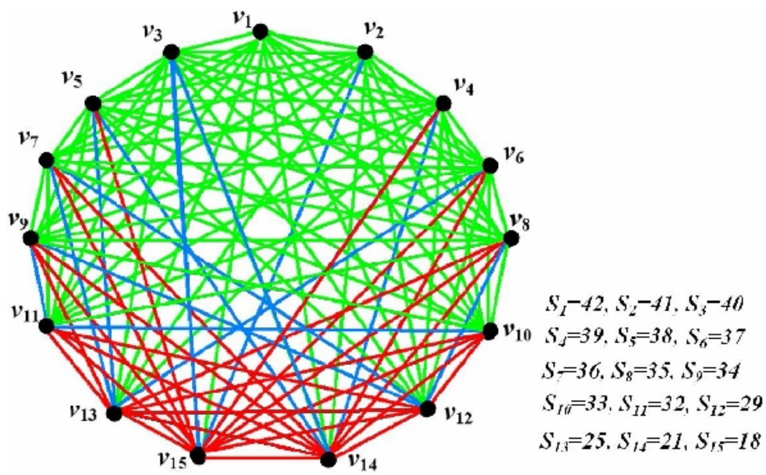

Figure 3. An example of algorithm to attain a 1-2-3-edge weighting and vertex coloring of K_{15}.

3. OPEN PROBLEMS AND CONJECTURES

In this section, we calculate some result that concluded from Theorem 1.1 and the algorithm for 1-2-3-edge weighting and vertex coloring of K_{3q}. In continue, we introduce some open problem and conjectures, which some of them maybe could be solved in the near future, by other reader.

Lemma 3.1. *Consider the 1-2-3-edge weighting and vertex coloring of K_{3q}= that obtained from above algorithm and using denotations of Definition 2.1, we have $\gamma_{3q}=\frac{3q^2-5q+2}{2}$, $\beta_{3q}=3q-2$ and $\alpha_{3q}=3q^2-2q+1$.*

Proof. The proving of lemma is easy; by refer to the algorithm for 1-2-3-edge weighting and vertex coloring of K_{3q} in proof of Theorem 1.1. Since obviously, q and $2q$ edges with weight 2 are inside to the vertices $v_2,v_4,\ldots,v_{2q}$ and $v_3,v_6,\ldots,v_{2q+1}$, respectively. Also, 3 edges with weight 2 are inside the vertex v_i ($i=2q+2,\ldots,3q-1$) and two edges with weight 2 are inside to v_{3q}. Thus $\beta_{3q}=\frac{q+2q+3(q-2)+2}{2}=3q-2$.□

Conjecture 3.2. (The 1-2-3-conjecture [6,8,10]) Every connected graph $G=(V,E)$ non-isomorph to K_2 (with at least two edges) has an edge weighting $f:E\rightarrow\{1,2,3\}$ and vertex coloring $s:V\rightarrow\{n-1,...,3n-3\}$.

Conjecture 3.3. (n vertex coloring) There are distinct numbers of S_v's, $v\in V(G)$ of a graph G of order n, for a 1-2-3-edge weighting and vertex coloring.

Conjecture 3.4. (The 1,2-conjecture [7,11,12]) Every graph G has a coloring chip configuration $c:V\cup E\rightarrow\{1,2\}$.

Conjecture 3.5. (Antimagic weighting [3,5]) For every connected graph G (with at least two edges) there is a bijection $c:E\rightarrow\{1,2,\ldots,|E|\}$ such that no two vertices of G have the same potential.

Conjecture 3.6. (Proper vertex coloring) For all graph G of order n, there are $\chi(G)$ numbers of S_v's, $v\in V(G)$ with this 1-2-3-edge weighting and vertex coloring. Where $\chi(G)$ is the number colors of the vertices on the graph G.

Conjecture 3.7. (Lucky weighting, [3,4]) For every graph G, there is a vertex weighting $c:V\rightarrow\{1,2,...,\chi(G)\}$, whose vertex potential $q_v=\sum_{u\in N(v)}c(u)$ is a proper coloring of G.

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