SUCCESSIVE LINEARIZATION SOLUTION OF A BOUNDARY LAYER CONVECTIVE HEAT TRANSFER OVER A FLAT PLATE

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ABSTRACT

The purpose of this paper is to discuss the flow of forced convection over a flat plate. The governing partial differential equations are transformed into ordinary differential equations using suitable transformations. The resulting equations were solved using a recent semi-numerical scheme known as the successive linearization method (SLM). A comparison between the obtained results with homotopy perturbation method and numerical method (NM) has been included to test the accuracy and convergence of the method.

KEYWORDS

Successive linearization method (SLM), Homotopy perturbation method, Forced convection.

1.INTRODUCTION

Many problems in fluid flow and heat transfer of boundary layers have attracted considerable attention in the last decades. Most of these problems are inherently of nonlinearity and they do not have analytical solution. Therefore, these nonlinear problems should be solved using other numerical methods. The solution of some nonlinear equations can be found using numerical techniques and some of them are solved using analytical methods such as Homotopy Perturbation Method (HPM). This problem was proposed by Ji-Huan He [1] and it has been applied to find a solution of nonlinear complicated engineering problems that cannot be solved by the known analytical methods. Cai et al. [2], Cveticanin [3], and El-Shahed [4] have been applied this method on integro-differential equations, Laplace transform, and fluid mechanics. Recently, there are many different methods have introduced some ways to obtain analytical solution for these nonlinear problems, such as the Homotopy Analysis Method (HAM) by Liao [5, 6], the Adomian decomposition method (ADM) [7, 8, 9], the variational iteration method (VIM) by He [10], the Differential Transformation Method by Zhou [11], Spectral Homotopy Analysis Method (SHAM) by Motsa et al. [12] and recently a novel successive linearization method (SLM) which has been used in a limited number of studies (see [13, 14, 15, 16, 17]) and it is used to solve the governing coupled non-linear system of equations. Recently [18, 19, 20] have reported that the SLM is more accurate and converges rapidly to the exact solution compared to other analytical techniques such as the Adomian decomposition method, homotopy perturbation method and variation iteration methods. Some of these methods, we should exert the small parameter in the equation. Therefore, finding the small parameters and exerting it in the equation are deficiencies of these techniques. The SLM method can be used in instead of traditional numerical methods such as Runge-Kutta, shooting methods, finite differences and finite elements in solving high non-linear differential equations. In this paper, we apply the Successive linearization method (SLM) to solve the problem of boundary layer convective heat transfer over a horizontal flat plate. The obtained results are compared with previous studies [21, 22, 23, 24, 25].

2.GOVERNING EQUATIONS

Let us consider the unsteady two-dimensional laminar flow of a viscous incompressible fluid. Under the boundary layer assumptions, the continuity and Navier-Stokes equations are [26]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}+g\beta\left(T-T_\infty\right), \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

In the above equations, u and v are the components of fluid velocity in the x and y directions respectively, ρ is the density of fluid, T is the fluid temperature, β is the coefficient of thermal expansion, g is the magnitude of acceleration due to gravity, ν is the kinematic viscosity and α is the specific heat. The initial and boundary conditions for this problem are

$$u=v=0, T=T_w \text{ at } y=0;\ u=U_\infty, T=T_\infty \text{ at } x=0$$

$$u\to U_\infty, T\to T_\infty \text{ as } y\to\infty;$$

Introducing:

$$\eta=\frac{y}{\sqrt{x}}\mathrm{Re}_x^{0.5}, \tag{4}$$

$$\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \tag{5}$$

where θ is a non-dimensional form of the temperature and the Reynolds number Re is defined as:

$$\mathrm{Re}=\frac{u_\infty x}{\nu}. \tag{6}$$

Using equations (1)-(5), the partial differential equations can be reduced to the following ordinary differential equations

$$f'''+\frac{1}{2}ff''=0, \tag{7}$$

$$\frac{1}{\mathrm{Pr}}\theta''+\frac{1}{2}f\theta'=0, \tag{8}$$

where f is related to the u velocity by

$$f'=\frac{u}{u_\infty}. \tag{9}$$

The transformed boundary conditions for the momentum and energy equations are [27]:

$$f(0)=0,\quad f'(0)=0,\quad \theta(0)=1,\quad f'(\infty)=1,\quad \theta(\infty)=0. \tag{10}$$

3.METHOD OF SOLUTION

The system of equations (7) and (8) together with the boundary conditions (10) were solved using a successive linearization method (SLM) (see [28, 29, 30]). The procedure of SLM is assumed that the unknown functions $f(\eta)$ and $\theta(\eta)$ can be written as

$$f(\eta) = f_i(\eta) + \sum_{m=0}^{i-1} F_m(\eta), \qquad \theta(\eta) = \theta_i(\eta) + \sum_{m=0}^{i-1} \Theta_m(\eta), \tag{11}$$

where F_m and Θ_m $(m \geq 1)$ are approximations which are obtained by solving the linear terms of the system of equations that obtained from substituting (11) in the ordinary differential equations (7) and (8). The main assumption of the SLM is that f_i and θ_i are very small when i becomes large, then nonlinear terms in f_i and θ_i and their derivatives are considered to be very small and therefore neglected. The initial guesses $F_0(\eta)$ and $\Theta_0(\eta)$ which are chosen to satisfy the boundary conditions

$$F_0(0) = 0, \quad F_0'(0) = 0, \quad \Theta_0(0) = 1, \quad F_0'(\infty) = 1, \quad \Theta_0(\infty) = 0, \tag{12}$$

which are taken to be

$$F_0(\eta) = \eta + e^{-\eta} - 1, \quad \Theta_0(\eta) = e^{-\eta}. \tag{13}$$

We start from the initial guesses $F_0(\eta)$ and $\Theta_0(\eta)$, the iterative solutions F_i and Θ_i are obtained by solving the resulting of linearized equations. The linearized system to be solved is

$$F_i''' + a_{1,i-1} F_i'' + a_{2,i-1} F_i = r_{1,i-1}, \tag{14}$$

$$b_{1,i-1} F_i + b_{2,i-1} \Theta_i'' + b_{3,i-1} \Theta_i' = r_{2,i-1}, \tag{15}$$

together with the boundary conditions

$$F_i(0) = F_i'(0) = \Theta_i(\infty) = 0, F_i'(\infty) = \Theta_i(0) = 1, \tag{16}$$

where

$$a_{1,i-1} = \frac{1}{2}\sum_{m=0}^{i-1} F_m, \quad a_{2,i-1} = \frac{1}{2}\sum_{m=0}^{i-1} F_m'', \quad b_{1,i-1} = \frac{1}{2}\sum_{m=0}^{i-1} \Theta_m', \quad b_{2,i-1} = \frac{1}{\Pr}, \quad b_{3,i-1} = \frac{1}{2}\sum_{m=0}^{i-1} F_m,$$

$$r_{1,i-1} = -\sum_{m=0}^{i-1} F_m''' - \frac{1}{2}\sum_{m=0}^{i-1} F_m \sum_{m=0}^{i-1} F_m'', \qquad r_{2,i-1} = -\frac{1}{\Pr}\sum_{m=0}^{i-1} \Theta_m'' - \frac{1}{2}\sum_{m=0}^{i-1} F_m \sum_{m=0}^{i-1} \Theta_m'.$$

The solutions of F_i and Θ_i, $i \geq 1$ can be found iteratively by solving equations (7) and (8). Finally, the solutions for $f(\eta)$ and $\theta(\eta)$ can be written as

$$f(\eta) \approx \sum_{m=0}^{M} F_m(\eta), \quad \theta(\eta) \approx \sum_{m=0}^{M} \Theta_m(\eta), \tag{17}$$

where M is termed the order of approximation. Equations (7) and (8) are solved using the Chebyshev spectral method which is based on the Chebyshev polynomials defined on the region $[-1,1]$. We have to transform the domain of solution $[0,\infty)$ into the region $[-1,1]$ where the problem is solved in the interval $[0,L]$ where L is a scale parameter used to invoke the boundary conditions at infinity. Thus, by using the mapping

$$\frac{\eta}{L}=\frac{\xi+1}{2}, \quad -1\le\xi\le 1. \tag{18}$$

The Gauss-Lobatto collocation points ξ_j is given by

$$\xi_j=\cos\frac{\pi j}{N}, \quad j=0,1,2,\ldots,N, \tag{19}$$

The functions F_i and Θ_i are approximated at the collocation points as

$$F_i(\xi)\approx\sum_{k=0}^{N}F_i(\xi_k)T_k(\xi_j), \quad \Theta_i(\xi)\approx\sum_{k=0}^{N}\Theta_i(\xi_k)T_k(\xi_j), \quad j=0,1,\ldots,N, \tag{20}$$

where T_k is the kth Chebyshev polynomial defined by

$$T_k(\xi)=\cos\left[k\cos^{-1}(\xi)\right]. \tag{21}$$

and

$$\frac{d^rF_i}{d\eta^r}=\sum_{k=0}^{N}\mathbf{D}_{kj}^r F_i(\xi_k), \quad \frac{d^r\Theta_i}{d\eta^r}=\sum_{k=0}^{N}\mathbf{D}_{kj}^r\Theta_i(\xi_k), \quad j=0,1,\ldots,N, \tag{22}$$

where r is the order of differentiation and $\mathbf{D}=\frac{2}{L}D$ with D being the Chebyshev spectral differentiation matrix ([31, 32, 33]), whose elements are defined as

$$\left.\begin{aligned}
D_{00}&=\frac{2N^2+1}{6},\\
D_{jk}&=\frac{c_j}{c_k}\frac{(-1)^{j+k}}{\xi_j-\xi_k}, \quad j\ne k; \quad j,k=0,1,\ldots,N,\\
D_{kk}&=-\frac{\xi_k}{2\left(1-\xi_k^2\right)}, \quad k=1,2,\ldots,N-1,\\
D_{NN}&=-\frac{2N^2+1}{6}.
\end{aligned}\right\} \tag{23}$$

Substitute (18)-(22) into equations (14) and (15) gives the matrix equation

$$\mathbf{A}_{i-1}\mathbf{X}_i=\mathbf{R}_{i-1}. \tag{24}$$

where $\mathbf{A}_{i-1}$ is a $(2N+2)\times(2N+2)$ square matrix and $\mathbf{X}_i$ and $\mathbf{R}_{i-1}$ are $(2N+2)\times 1$ column vectors given by

$$\mathbf{A}_{i-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \mathbf{X}_i = \begin{bmatrix} F_i \\ \Theta_i \end{bmatrix}, \quad \mathbf{R}_{i-1} = \begin{bmatrix} \mathbf{r}_{1,i-1} \\ \mathbf{r}_{2,i-1} \end{bmatrix}, \tag{25}$$

where

$$F_i = \left[f_i(\xi_0), f_i(\xi_1), \ldots, f_i(\xi_{N-1}), f_i(\xi_N) \right]^T,$$

$$\Theta_i = \left[\theta_i(\xi_0), \theta_i(\xi_1), \ldots, \theta_i(\xi_{N-1}), \theta_i(\xi_N) \right]^T,$$

$$\mathbf{r}_{1,i-1} = \left[r_{1,i-1}(\xi_0), r_{1,i-1}(\xi_1), \ldots, r_{1,i-1}(\xi_{N-1}), r_{1,i-1}(\xi_N) \right]^T,$$

$$\mathbf{r}_{2,i-1} = \left[r_{2,i-1}(\xi_0), r_{2,i-1}(\xi_1), \ldots, r_{2,i-1}(\xi_{N-1}), r_{2,i-1}(\xi_N) \right]^T,$$

$$A_{11} = \mathbf{D}^3 + \mathbf{a}_{1,i-1}\mathbf{D}^2 + \mathbf{a}_{2,i-1}\mathbf{I},$$

$$A_{12} = \mathbf{O},$$

$$A_{21} = \mathbf{b}_{1,i-1}\mathbf{I},$$

$$A_{22} = \mathbf{b}_{2,i-1}\mathbf{D}^2 + \mathbf{b}_{3,i-1}\mathbf{D}.$$

where T stands for transpose, $\mathbf{a}_{k,i-1}(k=1,2)$, $\mathbf{b}_{k,i-1}(k=1,2,3)$, and $\mathbf{r}_{k,i-1}(k=1,2)$ are diagonal matrices, $\mathbf{I}$ is the identity matrix, and $\mathbf{O}$ is the zero. Finally, the solution is given by

$$\mathbf{X}_i = \mathbf{A}_{i-1}^{-1}\mathbf{R}_{i-1}. \tag{26}$$

4. RESULTS AND DISCUSSION

The non-linear differential equations (7) and (8) together with the conditions (10) have been solved by using the SLM. We have taken $\eta_\infty = L = 15$, $N = 60$ for the implementation of SLM which gave sufficient accuracy. In order to validate our method, we have compared in Table 1 between the present results of $f'(\eta)$ and $\theta(\eta)$ corresponding to different values of η with those obtained by Adomian Decomposition Method (ADM) [25], Homotopy Perturbation Method (HPM) [24], and numerical method (NM) [21]. The results obtained by SLM are in excellent agreement with a few order SLM series giving accuracy of up to six decimal places. In Figures 1 to 3 comparison is made between our results, HPM [23,24] and NM [21] methods. It is clear from Figure 4 that, the temperature decreases with the increase in Prandtl number.

Table 1. The results of HPM, SLM, and NM methods for $f'(\eta)$ and $\theta(\eta)$.

η	$f'(\eta)$			$\theta(\eta)$		
	HPM	**SLM**	**NM**	**HPM**	**SLM**	**NM**
0	0	0	0	1	1	1
0.2	0.069907	0.066408	0.066408	0.930093	0.933592	0.933592
0.4	0.139764	0.132764	0.132764	0.860236	0.867236	0.867236
0.6	0.209441	0.198937	0.198937	0.790559	0.801063	0.801063
0.8	0.278723	0.264709	0.264709	0.721277	0.735291	0.735291

1.0	0.347312	0.329780	0.329780	0.652688	0.670220	0.670220
1.2	0.414831	0.393776	0.393776	0.585169	0.606224	0.606224
1.4	0.480832	0.456262	0.456262	0.519168	0.543738	0.543738
1.6	0.544806	0.516757	0.516757	0.455194	0.483243	0.483243
1.8	0.606195	0.574758	0.574758	0.393805	0.425242	0.425242
2.0	0.664414	0.629765	0.629766	0.335586	0.370235	0.370234
2.2	0.718871	0.681310	0.681310	0.281129	0.318690	0.318690
2.4	0.768993	0.728982	0.728982	0.231007	0.271018	0.271018
2.6	0.814261	0.772455	0.772455	0.185739	0.227545	0.227545
2.8	0.854239	0.811509	0.811510	0.145761	0.188491	0.188490
3.0	0.888611	0.846044	0.846044	0.111389	0.153956	0.143955
3.2	0.917222	0.876081	0.876081	0.082778	0.123919	0.123918
3.4	0.940107	0.901761	0.901761	0.059893	0.098239	0.088239
3.6	0.957524	0.923329	0.923330	0.042476	0.076671	0.066670
3.8	0.969974	0.941118	0.941118	0.030026	0.058882	0.058882
4.0	0.978212	0.955518	0.955518	0.021788	0.044482	0.031482
4.2	0.983235	0.966957	0.966957	0.016765	0.033043	0.033043
4.4	0.986244	0.975871	0.975871	0.013756	0.024129	0.024129
4.6	0.988579	0.982683	0.982684	0.011421	0.017317	0.017317
4.8	0.991602	0.987789	0.987790	0.008398	0.012211	0.012211
5.0	0.996533	0.991542	0.991542	0.003467	0.008458	0.008458

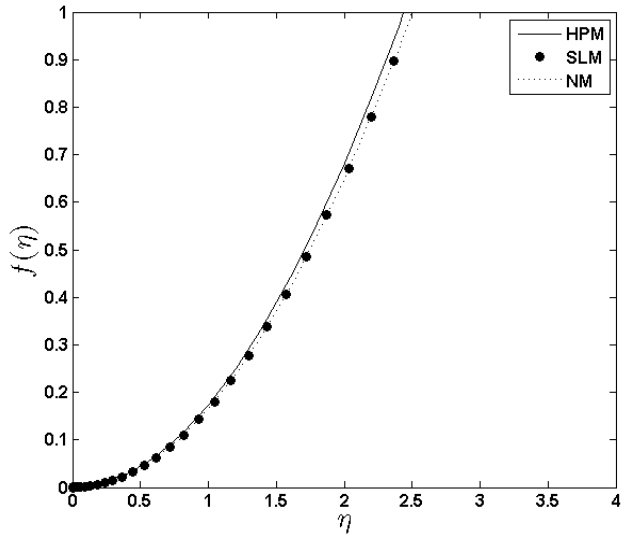

Figure 1. The comparison of the answers resulted by HPM [23], SLM, and NM for $f(\eta)$

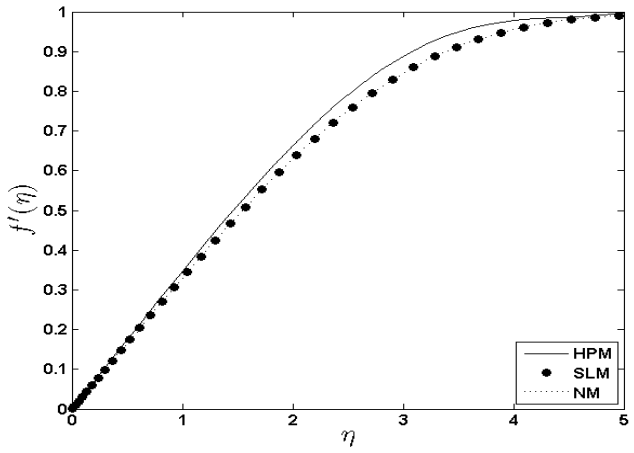

Figure 2. The comparison of the answers resulted by HPM [23], SLM, and NM for $f'(\eta)$

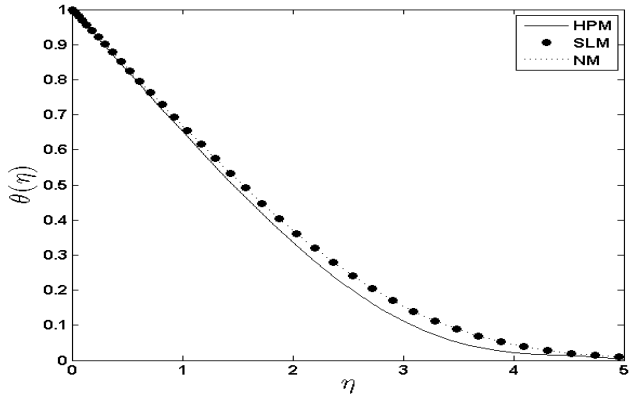

Figure 3. The comparison of the answers resulted by HPM [23], SLM, and NM for $\theta(\eta)$

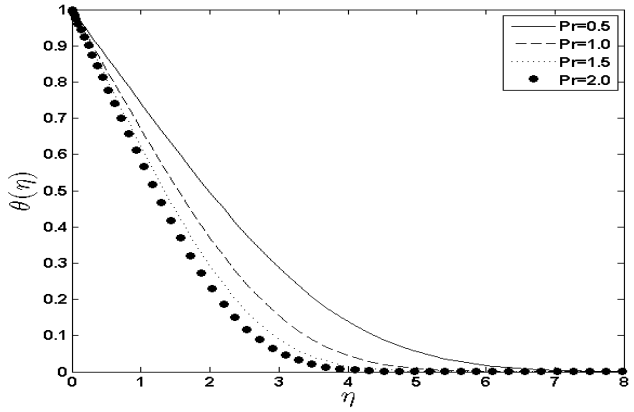

Figure 4. Effect of the Prandtl number $\Pr$ on $\theta(\eta)$

5.CONCLUSION

In this article, the SLM has been successfully applied to solve the problem of convective heat transfer. The partial differential equations are reduced into ordinary differential equations using similarity transformations. The present results indicate that this new method gives excellent approximations to the solution of the nonlinear equations and high accuracy compared to the other methods in solving non-linear differential equations. From the obtained results in the study, it was found that the temperature profile generally decreases with an increase in the values of the Prandtl number.

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