Interval-valued intuitionistic fuzzy closed ideals of BG-algebra and their products

Tapan Senapati*, Monoranjan Bhowmik#, Madhumangal Pal#

*Department of Mathematics, V. T. T. College, Midnapore- 721 101, India.
#Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore -721 102, India.

ABSTRACT

In this paper, we apply the concept of an interval-valued intuitionistic fuzzy set to ideals and closed ideals in BG-algebras. The notion of an interval-valued intuitionistic fuzzy closed ideal of a BG-algebra is introduced, and some related properties are investigated. Also, the product of interval-valued intuitionistic fuzzy BG-algebra is investigated.

KEYWORDS AND PHRASES

BG-algebras, interval-valued intuitionistic fuzzy sets (IVIFs), IVIF-ideals, IVIFC-ideals, homomorphism, equivalence relation, upper(lower)-level cuts, product of BG-algebra.

1. INTRODUCTION

Algebraic structures play an important role in mathematics with wide range of applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory etc.. On the other hand, in handling information regarding various aspects of uncertainty, non-classical logic (a great extension and development of classical logic) is considered to be more powerful technique than the classical logic one. The non-classical logic, therefore, has now a days become a useful tool in computer science. Moreover, non-classical logic deals with the fuzzy information and uncertainty. In 1965, Zadeh [28] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty in real physical world. Extending the concept of fuzzy sets (FSs), many scholars introduced various notions of higher-order FSs. Among them, interval-valued fuzzy sets (IVFSs) provides with a flexible mathematical framework to cope with imperfect and imprecise information. Moreover, Attanssova [2,6] introduced the concept of intuitionistic fuzzy sets (IFSs) and the interval-valued intuitionistic fuzzy sets (IVIFSs), as a generalization of an ordinary FSs.

In 1966, Imai and Iseki [13] introduced two classes of abstract algebra: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. In [11, 12] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebra is a proper subclass of the BCH-algebras. Neggers and...

In this paper, interval-valued intuitionistic fuzzy ideal (IVIF-ideal) of BG-subalgebras is defined. A lot of properties are investigated. The notion of equivalence relations on the family of all interval-valued intuitionistic fuzzy ideals of a BG-algebra is introduced and investigated some related properties. The product of (IVIF) BG-subalgebra has been introduced and some important properties of it are also studied.

The rest of this paper is organized as follows. The following section briefly reviews some background on BG-algebra, BG-subalgebra, refinement of unit interval, (IVIF) BG-subalgebra. In Section 3, the concepts and operations of (IVIF-ideal) and interval-valued intuitionistic fuzzy closed ideal (IVIFC-ideal) are proposed and discuss their properties in detail. In Section 4, some properties of IVIF-ideals under homomorphisms are investigated. In Section 5, equivalence relations on IVIF-ideals is introduced. In section 6, product of IVIF BG-subalgebra and some of its properties are studied. Finally, in Section 7, conclusion and scope of for future research are given.

2. PRELIMINARIES

In this section, some definitions are recalled which are used in the later sections. The BG-algebra is a very important branch of a modern algebra, which is defined by Kim and Kim [17]. This algebra is defined as follows.

Definition 1 [17] (BG-algebra) A non-empty set X with a constant 0 and a binary operation * is said to be BG-algebra if it satisfies the following axioms
1. = 0
2. x 0 = x
3. (x * y) * (0 * y) = x, for all x, y ∈ X.

A BG-algebra is denoted by (X,*,0). An example of BG-algebra is given below.

Example 1 Let X = {0,1,2,3,4,5} be a set. The binary operation * over X is defined as

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This table satisfies all the conditions of Definition 1. Hence, (X,*,0) is a BG-algebra.

A partial ordering ‘≤’ on X can be defined by x ≤ y if and only if x * y = 0. Now, we
introduce the concept of BG-subalgebra over a crisp set $X$ and the binary operation $*$ in the following.

**Definition 2** [17] (BG-subalgebra) A non-empty subset $S$ of a BG-algebra $X$ is called a subalgebra of $X$ if $x^*y \in S$, for all $x, y \in S$.

From this definition it is observed that, if a subset $S$ of a BG-algebra satisfies only the closer property, then $S$ becomes a BG-subalgebra.

**Definition 3** (Ideal) A non-empty subset $I$ of a BG-algebra $X$ is called an ideal of $X$ if $Ix \in I \quad \text{and} \quad Iy \in I \Rightarrow [xy \in I \quad \text{for any} \quad x, y \in X$.

An ideal $I$ of a BG-algebra $(X,*,0)$ is called closed if $Ix \in I \quad \text{for all} \quad Ix \in I$.

The IVIFS is a particular type of FS. Ahn and Lee [1] extends the concepts of BG-subalgebra from crisp set to fuzzy set. In the fuzzy set, the membership values of the elements are written together along with the elements. The membership values lie between 0 and 1. The definition of this set is given below.

**Definition 4** [28] (Fuzzy set) Let $X$ be the collection of objects denoted generally by $x$ then a fuzzy set $A$ in $X$ is defined as $A = \{<x, \mu_A(x) : x \in X \}$ where $\mu_A(x)$ is called the membership value of $x$ in $A$ and $0 \leq \mu_A(x) \leq 1$.

Combined the definition of BG-subalgebra over crisp set and the idea of fuzzy set Ahn and Lee [1] defined fuzzy BG-subalgebra, which is defined below.

**Definition 5** [1] (Fuzzy BG-subalgebra) Let $A$ be a fuzzy set in a BG-algebra. Then $A$ is called a fuzzy subalgebra of $X$ if $A(x^*y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in X$, where $\mu_A(x)$ is the membership value of $x$ in $A$.

**Definition 6** [19] (Fuzzy BG-ideal) A fuzzy set $A = \{<x, \mu_A(x) : x \in X \}$ in $X$ is called a fuzzy ideal of $X$ if it satisfies (i) $\mu_A(0) \geq \mu_A(x)$ and (ii) $\mu_A(x) \geq \min\{\mu_A(x^*y), \mu_A(y)\}$ for all $x, y \in X$.

In a fuzzy set only the membership value $\mu_A(x)$ of an element $x$ is considered, and the non-membership value can be taken as $1 - \mu_A(x)$. This value also lies between 0 and 1. But in reality this is not true for all cases, i.e., the non-membership value may be strictly less than 1. This idea was first incorporated by Attanasov [2] and initiated the concept of intuitionistic fuzzy set defined below.

**Definition 7** [2] (Intuitionistic fuzzy set) An intuitionistic fuzzy set $A$ over $X$ is an object having the form $A = \{<x, \mu_A(x), \nu_A(x) : x \in X \}$, where $\mu_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$, with the condition $0 \leq \nu_A(x) + \mu_A(x) \leq 1$ for all $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element $x$ in the set $A$. Obviously, when $\nu_A(x) = 1 - \mu_A(x)$ for every $x \in X$, the set $A$ becomes a fuzzy set.

Extending the idea of fuzzy BG-subalgebra, Zarandi and Saeid [31] defined intuitionistic fuzzy BG-subalgebra. In intuitionistic fuzzy BG-subalgebra, two conditions are to be satisfied, instead of one condition in fuzzy BG-subalgebra.

**Definition 8** [27] (Intuitionistic fuzzy BG-subalgebra) An IFS $A = \{<x, \mu_A(x), \nu_A(x) : x \in X \}$ in $X$ is called an intuitionistic fuzzy subalgebra of $X$ if it satisfies the following two conditions,
The people observed that the determination of membership value is a difficult task for a decision maker. In [29], Zadeh defined another type of fuzzy set called interval-valued fuzzy sets (IVFSs). The membership value of an element of this set is not a single number, it is an interval and this interval is a subinterval of the interval [0,1]. Let D[0,1] be the set of a subintervals of the interval [0,1].

**Definition 9** [29] (IVFS) An IVFS $A$ over $X$ is an object having the form $A = \{ (x, M_A(x)) : x \in X \}$, where $M_A : X \rightarrow D[0,1]$, and $D[0,1]$ is the set of all subintervals of [0,1]. The interval $M_A(x)$ denotes the interval of the degree of membership of the element $x$ to the set $A$, where $M_A(x) = [M_{M}(x), M_{AU}(x)]$ for all $x \in X$.

Combining the idea of intuitionistic fuzzy set and interval-valued fuzzy sets, Atanassov and Gargov [3] defined a new class of fuzzy set called interval-valued intuitionistic fuzzy sets (IVIFSs) defined below.

**Definition 10** [3] (IVIFS) An IVIFS $A$ over $X$ is an object having the form $A = \{ (x, M_A(x), N_A(x)) : x \in X \}$, where $M_A : X \rightarrow D[0,1]$ and $N_A : X \rightarrow D[0,1]$, where $D[0,1]$ is the set of all subintervals of [0,1]. The intervals $M_A(x)$ and $N_A(x)$ denote the intervals of the degree of membership and degree of non-membership of the element $x$ to the set $A$, where $M_A(x) = [M_{M}(x), M_{AU}(x)]$ and $N_A(x) = [N_{M}(x), N_{AU}(x)]$, for all $x \in X$, with the condition $0 \leq M_{AU}(x) + N_{AU}(x) \leq 1$.

Also note that $M_A(x) = [1 - M_{AU}(x), 1 - M_{M}(x)]$ and $N_A(x) = [1 - N_{AU}(x), 1 - N_{M}(x)]$, where $[M_A(x), N_A(x)]$ represents the complement of $x$ in $A$. For the sake of simplicity, we shall use the symbol $A = (M_A, N_A)$ for the IVIFS $A = \{ (x, M_A(x), N_A(x)) : x \in X \}$.

The determination of maximum and minimum between two real numbers is very simple, but it is not simple for two intervals. Biswas [7] described a method to find max/sup and min/inf between two intervals or a set of intervals.

**Definition 11** [7] (Refinement of intervals) Consider two elements $D_1, D_2 \in D[0,1]$. If $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$, then $rmax(D_1, D_2) = [\max(a_1, a_2), \max(b_1, b_2)]$ which is denoted by $D_1 \lor D_2$. Similarly, we $rmin(D_1, D_2) = [\min(a_1, a_2), \min(b_1, b_2)]$ which is denoted by $D_1 \land D_2$. Thus, if $D_i = [a_i, b_i] \in D[0,1]$ for $i=1,2,3,4,...$, then we define $r_{sup}(D_i) = [\sup(a_i), \sup(b_i)]$, i.e., $D_i = [\lor, a_i, \lor, b_i]$. Similarly, we $r_{inf}(D_i) = [\inf(a_i), \inf(b_i)]$ i.e., $D_i = [\land, a_i, \land, b_i]$. Now we call $D_i \geq D_2$ iff $a_i \geq a_2$ and $b_i \geq b_2$. Similarly, the relations $D_i \leq D_2$ and $D_i = D_2$ are defined.

The upper and lower level of an IVIF BG subalgebras is defined in the earlier paper of Senapati et al [28].

**Definition 12** [28](IVIF BG-subalgebras) Let $A = (M_A, N_A)$ be an IVIFS in $X$, where $X$ is a BG-subalgebra, then the set $A$ is IVIF BG-subalgebra over the binary operator $*$ if it satisfies the following conditions:

$$(\text{BGS1}) \quad M_A(x * y) \geq rmin\{M_A(x), M_A(y)\}$$
Definition 13 [28] Let \( A = (M_A, N_A) \) be an IVIF BG-subalgebra of \( X \). For \([s_i, s_j] \), \([t_i, t_j] \) \( \in [0,1] \), the set \( U(M_A : [s_i, s_j]) = \{ x \in X \mid M_A(x) \geq [s_i, s_j] \} \) is called upper \([s_i, s_j]-\)level of \( A \) and \( L(N_A : [t_i, t_j]) = \{ x \in X \mid N_A(x) \leq [t_i, t_j] \} \) is called lower \([t_i, t_j]-\)level of \( A \).

Also the mapping of an IVIFS is defined in [26]. It has some extensive properties in the field of IVIF BG-subalgebras.

Definition 14 [28] Let \( f \) be a mapping from a set \( X \) into a set \( Y \). Let \( B \) be an IVIFS in \( Y \). Then the inverse image of \( B \), i.e., \( f^{-1}(B) = (X, f^{-1}(M_B), f^{-1}(N_B)) \) is the IVIFS in \( X \) with the membership function and non-membership function respectively are given by \( f^{-1}(M_B)(x) = M_B(f(x)) \) and \( f^{-1}(N_B)(x) = N_B(f(x)) \).

3. IVIFC-IDEALS OF BG-ALGEBRAS

In this section, IVIF-ideal and IVIFC-ideal of BG-algebra are defined and prove some propositions and theorems are presented. In what follows, let \( X \) denote a BG-algebra unless otherwise specified.

Definition 15 An IVIFS \( A = (M_A, N_A) \) in \( X \) is called an IVIF-ideal of \( X \) if it satisfies:

\begin{align*}
\text{(BGS3)} & \quad M_A(0) \geq M_A(x) \quad \text{and} \quad N_A(0) \leq N_A(x) \\
\text{(BGS4)} & \quad M_A(x) \geq rmin\{M_A(x \ast y), M_A(y)\} \\
\text{(BGS5)} & \quad N_A(x) \leq rmax\{N_A(x \ast y), N_A(y)\}
\end{align*}

for all \( x, y \in X \).

Example 2 Consider a BG-algebra \( X = \{0,1,2,3\} \) with the following Cayley table

\[
\begin{array}{cccc}
* & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 2 & 1 & 0
\end{array}
\]

Let \( A = (M_A, N_A) \) be an IVIFS in \( X \) defined as \( M_A(0) = M_A(2) = [1,1] \), \( M_A(1) = M_A(3) = [m_1, m_2] \), \( N_A(0) = N_A(2) = [0,0] \) and \( N_A(1) = N_A(3) = [n_1, n_2] \), where \([m_1, m_2], [n_1, n_2] \in D[0,1] \) and \( m_2 + n_2 \leq 1 \). Then \( A = (M_A, N_A) \) is an IVIF-ideal of \( X \).

A closed ideal of IVIF ideal also be derived from the above definition.

Definition 16 An IVIFS \( A = (M_A, N_A) \) in \( X \) is called an IVIFC-ideal of \( X \) if it satisfies

\begin{align*}
\text{(BGS4), (BGS5) and (BGS6)} & \quad M_A(0 \ast x) \geq M_A(x) \quad \text{and} \quad N_A(0 \ast x) \leq N_A(x), \quad \text{for all} \quad x \in X.
\end{align*}

Example 3 Consider a BG-algebra \( X = \{0,1,2,3,4,5\} \) with the table in Example 1. We define an IVIFS \( A = (M_A, N_A) \) in \( X \) by, \( M_A(0) = [0.5,0.7] \), \( M_A(1) = M_A(2) = [0.4,0.6] \), \( M_A(3) = M_A(4) = M_A(5) = [0.3,0.4] \), \( N_A(0) = [0.1,0.2] \), \( N_A(1) = N_A(2) = [0.2,0.4] \), and \( N_A(3) = N_A(4) = N_A(5) = [0.4,0.6] \). By routine calculations, one can verify that \( A = (M_A, N_A) \) is an IVIFC-ideal of \( X \).
Proposition 1 Every IVIFC-ideal is an IVIF-ideal.

The converse of above proposition is not true in general as seen in the following example.

Example 4 Consider a BG-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the following table

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Let us an IVIFS $A = (M_A, N_A)$ in $X$ by $M_A(0) = [0.5, 0.7]$, $M_A(1) = [0.4, 0.6]$, $M_A(2) = M_A(3) = M_A(4) = M_A(5) = [0.3, 0.4]$, $N_A(0) = [0.1, 0.2]$, $N_A(1) = [0.2, 0.4]$, and $N_A(2) = N_A(3) = N_A(4) = N_A(5) = [0.4, 0.6]$. We know that $A = (M_A, N_A)$ is an IVIF-ideal of $X$. But it is not an IVIFC-ideal of $X$ since $M_A(0*0) \geq M_A(x)$ and $N_A(0*0) \leq N_A(x)$.

Corollary 1 Every IVIF BG-subalgebra satisfying (BGS4) and (BGS5) is an IVIF-ideal.

Theorem 1 Every IVIFC-ideal of a BG-algebra $X$ is an IVIF BG-subalgebra of $X$.

Proof: If $A = (M_A, N_A)$ is an IVIFC-ideal of $X$, then for any $x \in X$ we have $M_A(0*0) \geq M_A(x)$ and $N_A(0*0) \leq N_A(x)$. Now

$M_A(x*y) \geq rmin\{M_A((x*y)*(0*y)), M_A(0*0)\}$, by (BGS4)

$= rmin\{M_A(x), M_A(0*y)\}$

$\geq rmin\{M_A(x), M_A(y)\}$, by (BGS6)

and $N_A(x*y) \leq rmax\{N_A((x*y)*(0*y)), N_A(0*0)\}$, by (BGS5)

$= rmax\{N_A(x), N_A(0*y)\}$

$\leq rmax\{N_A(x), N_A(y)\}$, by (BGS6).

Hence the theorem.

Proposition 2 If an IVIFS $A = (M_A, N_A)$ in $X$ is an IVIFC-ideal, then for all $x \in X$, $M_A(0) \geq M_A(x)$ and $N_A(0) \leq N_A(x)$.

Proof: Straightforward.

Theorem 2 An IVIFSs $A = \{[M_{AL}, M_{AU}], [N_{AL}, N_{AU}]\}$ in $X$ is an IVIF-ideal of $X$ iff $M_{AL}$, $M_{AU}$, $N_{AL}$ and $N_{AU}$ are fuzzy ideals of $X$.

Proof: Since $M_{AL}(0) \geq M_{AL}(x)$, $M_{AU}(0) \geq M_{AU}(x)$, $N_{AL}(0) \leq N_{AL}(x)$ and $N_{AU}(0) \leq N_{AU}(x)$, therefore $M_A(0) \geq M_A(x)$ and $N_A(0) \leq N_A(x)$.

Let $M_{AL}$ and $M_{AU}$ are fuzzy ideals of $X$. Let $x, y \in X$. Then

$M_A(x) = [M_{AL}(x), M_{AU}(x)]$

$\geq [\min\{M_{AL}(x*y), M_{AL}(y)\}, \min\{M_{AU}(x*y), M_{AU}(y)\}]$

$= rmin\{[M_{AL}(x*y), M_{AU}(x*y)], [M_{AL}(y), M_{AU}(y)]\}$

$= rmin\{M_A(x*y), M_A(y)\}$.

Let $N_{AL}$ and $N_{AU}$ are fuzzy ideals of $X$ and $x, y \in X$. Then
\[ N_A(x) = [N_{AL}(x), N_{AU}(x)] \]
\[ \leq \{\max\{N_{AL}(x \ast y), N_{AL}(y)\}, \max\{N_{AU}(x \ast y), N_{AU}(y)\}\} \]
\[ = \text{rmax}\{[N_{AL}(x \ast y), N_{AU}(x \ast y)], [N_{AL}(y), N_{AU}(y)]\} \]
\[ = \text{rmax}\{N_A(x \ast y), N_A(y)\}. \]

Hence, \( A = \{[M_{AL}, M_{AU}], [N_{AL}, N_{AU}]\} \) is an IVIF ideal of \( X \).

Conversely, assume that, \( A \) is an IVIF ideal of \( X \). For any \( x, y \in X \), we have
\[ [M_{AL}(x), M_{AU}(x)] = M_A(x) \]
\[ \geq \text{rmin}\{M_A(x \ast y), M_A(y)\} \]
\[ = \text{rmin}\{[M_{AL}(x \ast y), M_{AU}(x \ast y)], [M_{AL}(y), M_{AU}(y)]\} \]
\[ = \{\min\{M_{AL}(x \ast y), M_{AL}(y)\}, \min\{M_{AU}(x \ast y), M_{AU}(y)\}\} \]
and
\[ [N_{AL}(x), N_{AU}(x)] = N_A(x) \]
\[ \leq \text{rmax}\{N_A(x \ast y), N_A(y)\} \]
\[ = \text{rmax}\{[N_{AL}(x \ast y), N_{AU}(x \ast y)], [N_{AL}(y), N_{AU}(y)]\} \]
\[ = \{\max\{N_{AL}(x \ast y), N_{AL}(y)\}, \max\{N_{AU}(x \ast y), N_{AU}(y)\}\}. \]

Thus,
\[ M_{AL}(x) \geq \min\{M_{AL}(x \ast y), M_{AL}(y)\}, M_{AU}(x) \geq \min\{M_{AU}(x \ast y), M_{AU}(y)\}, \]
\[ N_{AL}(x) \leq \max\{N_{AL}(x \ast y), N_{AL}(y)\}, N_{AU}(x) \leq \max\{N_{AU}(x \ast y), N_{AU}(y)\}. \]

Hence, \( M_{AL}, M_{AU}, N_{AL} \) and \( N_{AU} \) are fuzzy ideals of \( X \).

The intersection of two IVIFSs of \( X \) is defined by Atanassov \[4\] as follows

**Definition 17** Let \( A \) and \( B \) be two IVIFSs on \( X \), where
\[ A = \{[M_{AL}(x), M_{AU}(x)], [N_{AL}(x), N_{AU}(x)] : x \in X\} \]
and \( B = \{[M_{BL}(x), M_{BU}(x)], [N_{BL}(x), N_{BU}(x)] : x \in X\} \)

Then the intersection of \( A \) and \( B \) is denoted by \( A \cap B \) and is given by
\[ A \cap B = \{\langle x, M_{A \cap B}(x), N_{A \cap B}(x) \rangle : x \in X\} \]
\[ = \{[\min(M_{AL}(x), M_{BL}(x)), \min(M_{AU}(x), M_{BU}(x))], \]
\[ [\max(N_{AL}(x), N_{BL}(x)), \max(N_{AU}(x), N_{BU}(x))] : x \in X\} \]

The definition of intersection holds good for IVIF BG subalgebras.

**Theorem 3** Let \( A_1 \) and \( A_2 \) be two IVIF-ideals of a BG-algebras \( X \). Then \( A_1 \cap A_2 \) is also an IVIF-ideal of BG-algebra \( X \).

**Proof:** Let \( x, y \in A_1 \cap A_2 \). Then \( x, y \in A_1 \) and \( A_2 \). Now,
\[ M_{A \cap A_2}(0) = M_{A \cap A_2}(x \ast x) \geq \text{rmin}\{M_{A \cap A_2}(x), M_{A \cap A_2}(x)\} = M_{A \cap A_2}(x) \]
and
\[ N_{A \cap A_2}(0) = N_{A \cap A_2}(x \ast x) \leq \text{rmin}\{N_{A \cap A_2}(x), N_{A \cap A_2}(x)\} = N_{A \cap A_2}(x). \]

Also,
\[ M_{A \cap A_2}(x) = [M_{(A \cap A_2)}(x), M_{(A \cup A_2)}(x)] \]
\[ = [\min(M_{A \cap L}(x), M_{A \cap R}(x)), \min(M_{A \cup L}(x), M_{A \cup R}(x))]. \]

\[ \geq [\min(M_{(A \cap A_2)}(x \ast y), M_{(A \cap A_2)}(y)), \min(M_{(A \cup A_2)}(x \ast y), M_{(A \cup A_2)}(y))] \]
\[ = \text{rmin}\{M_{A \cap A_2}(x \ast y), M_{A \cap A_2}(y)\} \]
and
\[ N_{A \cap A_2}(x) = [N_{(A \cap A_2)}(x), N_{(A \cup A_2)}(x)]. \]
\[
\begin{align*}
= & \left[ \max(N_{A_1}(x), N_{A_2}(x)), \max(N_{A_1'}(x), N_{A_2'}(x)) \right] \\
\leq & \left[ \max(N_{(A_1 \cup A_2)_L}(x \ast y), N_{(A_1 \cup A_2)_R}(y)), \max(N_{(A_1 \cup A_2)_L'}(x \ast y), N_{(A_1 \cup A_2)_R'}(y)) \right] \\
= & \max\{N_{A_1 \cup A_2}(x \ast y), N_{A_1 \cup A_2}(y)\}.
\end{align*}
\]

Hence, \( A_1 \cap A_2 \) is also an IVIF-ideal of BG-algebra \( X \).

This proves that the intersection of any two IVIF-ideals of \( X \) is again an IVIF-ideal of \( X \). The above theorem can be generalized as

**Corollary 2** Intersection of any family of IVIF-ideals of \( X \) is again an IVIF-ideal of \( X \).

In the same way and by the definition of \( \tilde{A} \) we can prove the following result.

**Corollary 3** If \( A \) is an IVIF-ideal of \( X \) then \( \tilde{A} \) is also an IVIF-ideal of \( X \).

**Lemma 1** Let \( A = (M_A, N_A) \) be an IVIF-ideal of \( X \). If \( x \ast y \leq z \) then
\[
\begin{align*}
M_A(x) & \geq rmin\{M_A(y), M_A(z)\} \\
N_A(x) & \leq rmax\{N_A(y), N_A(z)\}.
\end{align*}
\]

**Proof:** Let \( x, y, z \in X \) such that \( x \ast y \leq z \). Then \( (x \ast y) \ast z = 0 \) and thus
\[
\begin{align*}
M_A(x) & \geq rmin\{M_A(x \ast y), M_A(y)\} \\
& \geq rmin\{rmin\{M_A(0), M_A(z)\}, M_A(y)\} \\
& = rmin\{rmin\{M_A(0), M_A(z)\}, M_A(y)\} \\
& = rmin\{M_A(y), M_A(z)\}
\end{align*}
\]
and
\[
\begin{align*}
N_A(x) & \leq rmax\{N_A(x \ast y), N_A(y)\} \\
& \leq rmax\{rmax\{N_A(0), N_A(z)\}, N_A(y)\} \\
& = rmax\{rmax\{N_A(0), N_A(z)\}, N_A(y)\} \\
& = rmax\{N_A(y), N_A(z)\}.
\end{align*}
\]

**Lemma 2** Let \( A = (M_A, N_A) \) be an IVIF-ideal of \( X \). If \( x \leq y \) then \( M_A(x) \geq M_A(y) \) and \( N_A(x) \leq N_A(y) \) i.e., \( M_A \) is order-preserving and \( N_A \) is order-preserving.

**Proof:** Let \( x, y \in X \) such that \( x \leq y \). Then \( x \ast y = 0 \) and thus
\[
\begin{align*}
M_A(x) & \geq rmin\{M_A(x \ast y), M_A(y)\} \\
& = rmin\{M_A(0), M_A(y)\} \\
& = M_A(x)
\end{align*}
\]
and
\[
\begin{align*}
N_A(x) & \leq rmax\{N_A(x \ast y), N_A(y)\} \\
& = rmax\{N_A(0), N_A(y)\} \\
& = N_A(x).
\end{align*}
\]

Using induction on \( n \) and by Lemma 1 and Lemma 2 we can easily prove the following theorem.

**Theorem 4** If \( A = (M_A, N_A) \) is an IVIF-ideal of \( X \), then \( \ldots((x \ast a_1) \ast a_2) \ast \ldots) \ast a_n = 0 \) for any \( x, a_1, a_2, \ldots, a_n \in X \), implies \( M_A(x) \geq rmin\{M_A(a_1), M_A(a_2), \ldots, M_A(a_n)\} \) and \( N_A(x) \leq rmax\{N_A(a_1), N_A(a_2), \ldots, N_A(a_n)\} \).
Here we define two operators $\bigoplus A$ and $\bigotimes A$ on IVIFS as follows:

**Definition 18** Let $A = (M_A, N_A)$ be an IVIFS defined on $X$. The operators $\bigoplus A$ and $\bigotimes A$ are defined as $\bigoplus A = (M_A(x), \overline{M}_A(x))$ and $\bigotimes A = (\overline{N}_A(x), N_A(x))$ in $X$.

**Theorem 5** If $A = (M_A, N_A)$ is an IVIF-ideal of a BG-algebra $X$, then (i) $\bigoplus A$, and (ii) $\bigotimes A$, both are IVIF-ideals of BG-algebra $X$.

**Proof:** For (i), it is sufficient to show that $\bigoplus A$ satisfies the second part of the conditions (BGS3) and (BGS5). We have

$$\overline{M}_A(x) = 1 - M_A(x) \leq 1 - \text{rmin}\{M_A(x \ast y), M_A(y)\} = \text{rmax}\{1 - M_A(x \ast y), 1 - M_A(y)\} \text{ since } 1 = [1,1]$$

$$= \text{rmax}\{\overline{M}_A(x \ast y), \overline{M}_A(y)\}.$$ 

Hence, $\bigoplus A$ is an IVIF-ideal of BG-subalgebra $X$.

For (ii), it is sufficient to show that $\bigotimes A$ satisfies the first part of the conditions (BGS3) and (BGS4). We have

$$\overline{N}_A(x) = 1 - N_A(x) \geq 1 - \text{rmax}\{N_A(x \ast y), N_A(y)\} = \text{rmin}\{1 - N_A(x \ast y), 1 - N_A(y)\} \text{ since } 1 = [1,1]$$

$$= \text{rmin}\{\overline{N}_A(x \ast y), \overline{N}_A(y)\}.$$ 

Hence, $\bigotimes A$ is an IVIF-ideal of BG-algebra $X$.

**Theorem 6** An IVIFS $A = (M_A, N_A)$ is an IVIFC-ideal of $X$ iff the sets $U(M_A : [s_1, s_2])$ and $L(N_A : [t_1, t_2])$ are closed ideal of $X$ for every $[s_1, s_2],[t_1, t_2] \in D[0,1]$.

**Proof:** Suppose that $A = (M_A, N_A)$ is an IVIFC-ideal of $X$. For $[s_1, s_2] \in D[0,1]$, obviously, $0 \ast x \in U(M_A : [s_1, s_2])$, where $x \in X$. Let $x, y \in X$ be such that $x \ast y \in U(M_A : [s_1, s_2])$ and $y \in U(M_A : [s_1, s_2])$. Then $M_A(x) \geq \text{rmin}\{M_A(x \ast y), M_A(y)\} \geq [s_1, s_2]$. Then $x \in U(M_A : [s_1, s_2])$. Hence, $U(M_A : [s_1, s_2])$ is closed ideal of $X$.

For $[t_1, t_2] \in D[0,1]$, obviously, $0 \ast x \in L(N_A : [t_1, t_2])$. Let $x, y \in X$ be such that $x \ast y \in L(N_A : [t_1, t_2])$ and $y \in L(N_A : [t_1, t_2])$. Then $N_A(x) \leq \text{rmax}\{N_A(x \ast y), N_A(y)\} \leq [t_1, t_2]$. Then $x \in L(N_A : [t_1, t_2])$. Hence, $L(N_A : [t_1, t_2])$ is closed ideal of $X$.

Conversely, assume that each non-empty level subset $U(M_A : [s_1, s_2])$ and $L(N_A : [t_1, t_2])$ are closed ideals of $X$. For any $x \in X$, let $M_A(x) = [s_1, s_2]$ and $N_A(x) = [t_1, t_2]$. Then $x \in U(M_A : [s_1, s_2])$ and $x \in L(N_A : [t_1, t_2])$. Since $0 \ast x \in U(M_A : [s_1, s_2]) \cap L(N_A : [t_1, t_2])$, it follows that $M_A(0 \ast x) \geq [s_1, s_2] = M_A(x)$ and $N_A(x) \leq [t_1, t_2] = N_A(x)$, for all $x \in X$.

If there exist $\alpha, \beta \in X$ such that $M_A(\alpha) < \text{rmin}\{M_A(\alpha \ast \beta), M_A(\beta)\}$, then by taking $[s_1', s_2'] = \frac{1}{2}[M_A(\alpha \ast \beta) + \text{rmin}\{M_A(\alpha), M_A(\beta)\}]$, it follows that $\alpha \ast \beta \in U(M_A : [s_1', s_2'])$. Therefore, $\bigoplus A$ and $\bigotimes A$ are IVIF-ideals of $X$.
and $\beta \in U(M_A;[s_1',s_2'])$, but $\alpha \not\in U(M_A;[s_1',s_2'])$, which is a contradiction. Hence, $U(M_A;[s_1',s_2'])$ is not closed ideal of $X$.

Again, if there exist $\gamma, \delta \in X$ such that $N_A(\gamma) > rmax\{N_A(\gamma \ast \delta), N_A(\delta)\}$, then by taking $[t_1',t_2'] = \frac{1}{2}[N_A(\gamma \ast \delta) + rmax\{N_A(\gamma), N_A(\delta)\}]$, it follows that $\gamma \ast \delta \in U(N_A;[t_1',t_2'])$ and $\delta \in L(N_A;[t_1',t_2'])$, but $\gamma \not\in L(N_A;[t_1',t_2'])$, which is a contradiction. Hence, $L(N_A;[t_1',t_2'])$ is not closed ideal of $X$.

Hence, $A = (M_A,N_A)$ is an IVFC-ideal of $X$ since it satisfies (BGS3) and (BGS4).

**4. INVESTIGATION OF IVIF-IDEALS UNDER HOMOMORPHISM**

In this section, homomorphism of IVIF BG-subalgebra is defined and some results are studied. Let $f$ be a mapping from the set $X$ into the set $Y$. Let $B$ be an IVIFS in $Y$. Then the inverse image of $B$, is defined as $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$ with the membership function and non-membership function respectively are given by $f^{-1}(M_B)(x) = M_B(f(x))$ and $f^{-1}(N_B)(x) = N_B(f(x))$. It can be shown that $f^{-1}(B)$ is an IVIFS.

**Definition 19** A mapping $f : X \to Y$ of BG-algebra is called a BG-homomorphism if $f(x \ast y) = f(x) \ast f(y)$, for all $x,y \in X$. Note that if $f : X \to Y$ is a BG-homomorphism, then $f(0) = 0$.

**Theorem 7** [28] Let $f : X \to Y$ be a homomorphism of BG-algebras. If $B = (M_B,N_B)$ is an IVIF BG-subalgebra of $Y$, then the preimage $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$ of $B$ under $f$ is an IVIF BG-subalgebra of $X$.

**Theorem 8** Let $f : X \to Y$ be a homomorphism of BG-algebras. If $B = (M_B,N_B)$ is an IVIF-ideal of $Y$, then the preimage $f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B))$ of $B$ under $f$ in $X$ is an IVIF-ideal of $X$.

**Proof:** For all $x \in X$, $f^{-1}(M_B)(x) = M_B(f(x)) \leq M_B(0) = M_B(f(0)) = f^{-1}(M_B)(0)$ and $f^{-1}(N_B)(x) = N_B(f(x)) \geq N_B(0) = N_B(f(0)) = f^{-1}(N_B)(0)$. Again let $x, y \in X$. Then $f^{-1}(M_B)(x) = M_B(f(x))$

$$\geq rmin\{M_B((f(x) \ast f(y)), M_B(f(y))\}$$

$$\geq rmin\{M_B(f(x \ast y), M_B(f(y))\}$$

$$= rmin\{f^{-1}(M_B)(x \ast y), f^{-1}(M_B)(y)\}$$

and $f^{-1}(N_B)(x) = N_B(f(x))$

$$\leq rmax\{N_B((f(x) \ast f(y)), N_B(f(y))\}$$

$$\leq rmax\{N_B(f(x \ast y), N_B(f(y))\}$$

$$= rmax\{f^{-1}(N_B)(x \ast y), f^{-1}(N_B)(y)\}.$$
Hence, \( f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B)) \) is an IVIF-ideal of \( X \).

**Theorem 9** Let \( f : X \to Y \) be an epimorphism of BG-algebras. Then \( B = (M_B, N_B) \) is an IVIF-ideal of \( Y \), if \( f^{-1}(B) = (f^{-1}(M_B), f^{-1}(N_B)) \) of \( B \) under \( f \) in \( X \) is an IVIF-ideal of \( X \).

**Proof:** For any \( x \in Y \), \( \exists \ a \in X \) such that \( f(a) = x \). Then
\[
M_B(a) = M_B(f(a)) = f^{-1}(M_B)(a) \leq f^{-1}(M_B)(0) = M_B(f(0)) = M_B(0)
\]
and
\[
N_B(a) = N_B(f(a)) = f^{-1}(N_B)(a) \geq f^{-1}(N_B)(0) = N_B(f(0)) = N_B(0).
\]
Let \( x, y \in Y \). Then \( f(a) = x \) and \( f(b) = y \) for some \( a, b \in X \). Thus
\[
M_B(x) = M_B(f(a)) = f^{-1}(M_B)(a) 
\]
\[
\geq rmin\{f^{-1}(M_B)(a*b), f^{-1}(M_B)(b)\} 
\]
\[
= rmin\{M_B(f(a*b)), M_B(f(b))\} 
\]
\[
= rmin\{M_B((x* y), M_B(y))\} 
\]
and
\[
N_B(x) = N_B(f(a)) = f^{-1}(N_B)(a) 
\]
\[
\leq rmax\{f^{-1}(N_B)(a*b), f^{-1}(N_B)(b)\} 
\]
\[
= rmax\{N_B(f(a*b)), N_B(f(b))\} 
\]
\[
= rmax\{N_B((x* y), N_B(y))\} 
\]
Then \( B = (M_B, N_B) \) is an IVIF-ideal of \( Y \).

### 5. EQUIVALENCE RELATIONS ON IVIF-IDEALS

Let \( IVIFI(X) \) denote the family of all interval-valued intuitionistic fuzzy ideals of \( X \) and let \( \rho = [\rho_1, \rho_2] \in D(0,1] \). Define binary relations \( U^\rho \) and \( L^\rho \) on \( IVIFI(X) \) as
\[(A, B) \in U^\rho \Leftrightarrow U(M_A; \rho) = U(M_B; \rho) \] and \( (A, B) \in L^\rho \Leftrightarrow L(N_A; \rho) = L(N_B; \rho) \)
respectively, for \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) in \( IVIFI(X) \). Then clearly \( U^\rho \) and \( L^\rho \) are equivalence relations on \( IVIFI(X) \). For any \( A = (M_A, N_A) \in IVIFI(X) \), let \( [A]_{U^\rho} \) (respectively, \( [A]_{L^\rho} \)) denote the equivalence class of \( A \) modulo \( U^\rho \) (respectively, \( L^\rho \)), and denote by \( IVIFI(X)/U^\rho \) (respectively, \( IVIFI(X)/L^\rho \)) the collection of all equivalence classes modulo \( U^\rho \) (respectively, \( L^\rho \)), i.e.,
\[
IVIFI(X)/U^\rho := \{[A]_{U^\rho} \mid A = (M_A, N_A) \in IVIFI(X)\},
\]
respectively,
\[
IVIFI(X)/L^\rho := \{[A]_{L^\rho} \mid A = (M_A, N_A) \in IVIFI(X)\}.
\]
These two sets are also called the quotient sets.

Now let \( T(X) \) denote the family of all ideals of \( X \) and let \( \rho = [\rho_1, \rho_2] \in D(0,1] \). Define mappings \( f_\rho \) and \( g_\rho \) from \( IVIFI(X) \) to \( T(X) \cup \{\phi\} \) by \( f_\rho(A) = U(M_A; \rho) \) and \( g_\rho(A) = L(N_A; \rho) \), respectively, for all \( A = (M_A, N_A) \in IVIFI(X) \). Then \( f_\rho \) and \( g_\rho \) are
clearly well-defined.

**Theorem 10** For any \( \rho = [\rho_1, \rho_2] \in D[0,1] \), the maps \( f_\rho \) and \( g_\rho \) are surjective from \( \text{IVIFI}(X) \) to \( T(X) \cup \{\emptyset\} \).

**Proof:** Let \( \rho = [\rho_1, \rho_2] \in D[0,1] \). Note that \( \tilde{\emptyset} = (0,1) \) is in \( \text{IVIFI}(X) \), where \( 0 \) and \( 1 \) are interval-valued fuzzy sets in \( X \) defined by \( 0(x) = [0,0] \) and \( 1(x) = [1,1] \) for all \( x \in X \). Obviously \( f_\rho(\tilde{\emptyset}) = U(0: \rho) = U([0,0]: [\rho_1, \rho_2]) = \phi = L([1,1]: [\rho_1, \rho_2]) = L(1: \rho) = g_\rho(\tilde{\emptyset}) \).

Let \( P(\not\in \phi) \in \text{IVIFI}(X) \). For \( \tilde{P} = (\chi_p, \overline{\chi}_p) \in \text{IVIFI}(X) \), we have \( f_\rho(\tilde{P}) = U(\chi_p : \rho) = P \) and \( g_\rho(\tilde{P}) = L(\overline{\chi}_p : \rho) = P \). Hence \( f_\rho \) and \( g_\rho \) are surjective.

**Theorem 11** The quotient sets \( \text{IVIFI}(X)/U^\rho \) and \( \text{IVIFI}(X)/L^\rho \) are equipotent to \( T(X) \cup \{\emptyset\} \) for every \( \rho \in D[0,1] \).

**Proof:** For \( \rho \in D[0,1] \) let \( f_\rho^* \) (respectively, \( g_\rho^* \)) be a map from \( \text{IVIFI}(X)/U^\rho \) (respectively, \( \text{IVIFI}(X)/L^\rho \)) to \( T(X) \cup \{\emptyset\} \) defined by \( f_\rho^*([A]_{U^\rho}) = f_\rho(A) \) (respectively, \( g_\rho^*([A]_{U^\rho}) = g_\rho(A) \)) for all \( A = (M_A, N_A) \in \text{IVIFI}(X) \). If \( U(M_A : \rho) = U(M_B : \rho) \) and \( L(N_A : \rho) = L(N_B : \rho) \) for \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) in \( \text{IVIFI}(X) \), then \( (A, B) \in U^\rho \) and \( (A, B) \in L^\rho \); hence \( [A]_{U^\rho} = [B]_{U^\rho} \) and \( [A]_{L^\rho} = [B]_{L^\rho} \). Therefore the maps \( f_\rho^* \) and \( g_\rho^* \) are injective. Now let \( P(\not\in \phi) \in \text{IVIFI}(X) \). For \( \tilde{P} = (\chi_p, \overline{\chi}_p) \in \text{IVIFI}(X) \), we have

\[
 f_\rho^*([\tilde{P}]_{U^\rho}) = f_\rho(\tilde{P}) = U(\chi_p : \rho) = P, \\
 g_\rho^*([\tilde{P}]_{L^\rho}) = g_\rho(\tilde{P}) = L(\overline{\chi}_p : \rho) = P.
\]

Finally, for \( \tilde{0} = (0,1) \in \text{IVIFI}(X) \) we get \( f_\rho^*([\tilde{0}]_{U^\rho}) = f_\rho(\tilde{0}) = U(0 : \rho) = \phi \) and \( g_\rho^*([\tilde{0}]_{L^\rho}) = g_\rho(\tilde{0}) = L(1 : \rho) = \phi \). This shows that \( f_\rho^* \) and \( g_\rho^* \) are surjective. This completes the proof.

For any \( \rho \in D[0,1] \), we define another relation \( R^\rho \) on \( \text{IVIFI}(X) \) as follows:

\[
 (A, B) \in R^\rho \iff U(M_A : \rho) \cap L(N_A : \rho) = U(M_B : \rho) \cap L(N_B : \rho)
\]

for any \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \in \text{IVIFI}(X) \). Then the relation \( R^\rho \) is an equivalence relation on \( \text{IVIFI}(X) \).

**Theorem 12** For any \( \rho \in D[0,1] \), the maps \( \psi_\rho : \text{IVIFI}(X) \rightarrow T(X) \cap \{\emptyset\} \) defined by \( \psi_\rho(A) = f_\rho(A) \cap g_\rho(A) \) for each \( A = (M_A, N_A) \in X \) is surjective.

**Proof:** Let \( \rho \in D[0,1] \). For \( \tilde{0} = (0,1) \in \text{IVIFI}(X) \),

\[
 \psi_\rho(\tilde{0}) = f_\rho(\tilde{0}) \cap g_\rho(\tilde{0}) = U(0 : \rho) \cap L(1 : \rho) = \phi.
\]

For any \( H \in \text{IVIFI}(X) \), there exists \( H = (\chi_H, \overline{\chi}_H) \in \text{IVIFI}(X) \) such that

\[
 \psi_\rho(H) = f_\rho(H) \cap g_\rho(H) = U(\chi_H : \rho) \cap L(\overline{\chi}_H : \rho) = H.
\]

This completes the proof.
Theorem 13 The quotient sets IVIFI(X)/Rρ are equipotent to T(X) ∪ {ϕ} for every ρ ∈ D[0,1].

Proof: For ρ ∈ D[0,1], define a map ψρ : IVIFI(X)/Rρ → T(X) ∪ {ϕ} by ψρ([A]ρρρ) = ψρ(A) for all [A]ρρρ ∈ IVIFI(X)/Rρ. Assume that ψρ([A]ρρρ) = ψρ([B]ρρρ) for any [A]ρρρ and [B]ρρρ ∈ IVIFI(X)/Rρ. Then fρ(A) ∩ gρ(A) = fρ(B) ∩ gρ(B), i.e.,

\[ U(M_A : ρ) \cap L(N_A : ρ) = U(M_B : ρ) \cap L(N_B : ρ). \]

Hence (A,B) ∈ Rρ, and so [A]ρρρ = [B]ρρρ. Therefore the maps ψρ are injective. Now for 0 = (0,1) ∈ IVIFI(X) we have

\[ \psi^*_ρ(0) = f_ρ(0) \cap g_ρ(0) = U(0 : ρ) \cap L(1 : ρ) = ϕ. \]

If H ∈ IVIFI(X), then for H : (χH - χH) ∈ IVIFI(X), we obtain

\[ \psi^*_ρ(H) = f_ρ(H) \cap g_ρ(H) = U(χH : ρ) \cap L(χH : ρ) = H. \]

Thus ψ^*_ρ is surjective. This completes the proof.

6. PRODUCT OF IVIF BG-ALGEBRA

In this section, product of IVIF BG-algebra is defined and some results are studied.

Definition 20 Let A = (M_A, N_A) and B = (M_B, N_B) be two IVIFSs of X. The cartesian product A × B = (X × X, M_A × M_B, N_A × N_B) is defined by

\[ (M_A \times M_B)(x,y) = rmin\{M_A(x), M_B(y)\} \]

and \[ (N_A \times N_B)(x,y) = rmax\{N_A(x), N_B(y)\}, \]

where \( M_A \times M_B : X \times X \to D[0,1] \) and \( N_A \times N_B : X \times X \to D[0,1] \) for all \( x, y \in X \).

Proposition 3 Let A = (M_A, N_A) and B = (M_B, N_B) be IVIF-ideals of X, then A × B is an IVIF-ideal of X × X.

Proof: For any \( (x, y) \in X \times X \), we have

\[ (M_A \times M_B)(0,0) = rmin\{M_A(0), M_B(0)\} \geq rmin\{M_A(x), M_B(y)\}, \text{ for all } x, y \in X, \]

\[ = (M_A \times M_B)(x,y) \]

and \[ (N_A \times N_B)(0,0) = rmax\{N_A(0), N_B(0)\} \leq rmin\{N_A(x), N_B(y)\}, \text{ for all } x, y \in X, \]

\[ = (N_A \times N_B)(x,y). \]

Let \( (x_1, y_1) \) and \( (x_2, y_2) \) ∈ X × X. Then

\[ (M_A \times M_B)(x_1, y_1) = rmin\{M_A(x_1), M_B(y_1)\} \]

\[ \geq rmin\{rmin\{M_A(x_1 \times x_2), M_A(x_2)\}, rmin\{M_B(y_1 \times y_2), M_B(y_2)\}\} \]

\[ = rmin\{rmin\{M_A(x_1 \times x_2), M_B(y_1 \times y_2)\}, rmin\{M_A(x_2), M_B(y_2)\}\} \]

\[ = rmin\{rmin\{M_A \times M_B(x_1 \times x_2, y_1 \times y_2), (M_A \times M_B)(x_2, y_2)\}\}. \]
\[ (N_A \times N_B)(x, y) = \max\{N_A(x), N_B(y)\} \]

and
\[ (N_A \times N_B)(x, y) = \min\{N_A(x), N_B(y)\} \]

Hence, \( A \times B \) is an IVIF-ideal of \( X \times X \).

**Proposition 4** Let \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIF-ideals of \( X \), then \( A \times B \) is an IVIF-ideal of \( X \times X \).

**Proof:** Now,
\[ (M_A \times M_B)((0, 0) \ast (x, y)) = (M_A \times M_B)(0 \ast x, 0 \ast y) \]
\[ = \min\{M_A(0 \ast x), M_B(0 \ast y)\} \]
\[ \geq \min\{M_A(x), M_B(y)\} \]
\[ = (M_A \times M_B)(x, y) \]

and
\[ (N_A \times N_B)((0, 0) \ast (x, y)) = (N_A \times N_B)(0 \ast x, 0 \ast y) \]
\[ = \max\{N_A(0 \ast x), N_B(0 \ast y)\} \]
\[ \leq \max\{N_A(x), N_B(y)\} \]
\[ = (N_A \times N_B)(x, y). \]

Hence, \( A \times B \) is an IVIF-ideal of \( X \times X \).

**Lemma 3** If \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIF-ideals of \( X \), then
\[ \oplus (A \times B) = (M_A \times M_B, \overline{M_A} \times \overline{M_B}) \]

is an IVIF-ideal of \( X \times X \).

**Proof:** Since \( (M_A \times M_B)(x, y) = \min\{M_A(x), M_B(y)\} \).

That is,
\[ 1 - (M_A \times M_B)(x, y) = \min\{1 - M_A(x), 1 - M_B(y)\}. \]

This implies,
\[ 1 - \min\{1 - M_A(x), 1 - M_B(y)\} = (\overline{M_A} \times \overline{M_B})(x, y). \]

Therefore,
\[ (\overline{M_A} \times \overline{M_B})(x, y) = \max\{\overline{M_A}(x), \overline{M_B}(y)\}. \]

Hence, \( \oplus (A \times B) = (M_A \times M_B, \overline{M_A} \times \overline{M_B}) \)

is an IVIF-ideal of \( X \times X \).

**Lemma 4** If \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIF-ideals of \( X \), then
\[ \otimes (A \times B) = (\overline{N_A} \times \overline{N_B}, N_A \times N_B) \]

is an IVIF-ideal of \( X \times X \).

**Proof:** Since \( (N_A \times N_B)(x, y) = \max\{N_A(x), N_B(y)\} \).

That implies,
\[ 1 - (N_A \times N_B)(x, y) = \max\{1 - N_A(x), 1 - N_B(y)\} \]

This is,
\[ 1 - \max\{1 - N_A(x), 1 - N_B(y)\} = (\overline{N_A} \times \overline{N_B})(x, y). \]

Therefore,
\[ (\overline{N_A} \times \overline{N_B})(x, y) = \min\{\overline{N_A}(x), \overline{N_B}(y)\}. \]

Hence, \( \otimes (A \times B) = (\overline{N_A} \times \overline{N_B}, N_A \times N_B) \)

is an IVIF-ideal of \( X \times X \).

By the above two lemmas, it is not difficult to verify that the following theorem is valid.
Theorem 14 The IVIFSs \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIF-ideals of \( X \) iff
\[
\bigoplus (A \times B) = (M_A \times M_B, M_A \times M_B) \quad \text{and} \quad \bigotimes (A \times B) = (N_A \times N_B, N_A \times N_B)
\]
are IVIF-ideal of \( X \times X \).

Lemma 5 If \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIFC-ideals of \( X \), then
\[
\bigoplus (A \times B) = (M_A \times M_B, M_A \times M_B)
\]
is an IVIFC-ideal of \( X \times X \).

Proof: Since \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIFC-ideals of \( X \), \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIF-ideals of \( X \). Thus, \( A \times B \) is IVIF-ideal of \( X \times X \).

Now \( (M_A \times M_B)((0,0)*(x,y)) \geq (M_A \times M_B)(x,y) \).

That is, \( 1 - (M_A \times M_B)((0,0)*(x,y)) \leq 1 - (M_A \times M_B)(x,y) \).

This gives, \( (M_A \times M_B)((0,0)*(x,y)) \leq (M_A \times M_B)(x,y) \).

Hence, \( \bigoplus (A \times B) = (M_A \times M_B, M_A \times M_B) \) is an IVIFC-ideal of \( X \times X \).

Lemma 6 If \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIFC-ideals of \( X \), then
\[
\bigotimes (A \times B) = (N_A \times N_B, N_A \times N_B)
\]
is an IVIFC-ideals of \( X \times X \).

Proof: The proof is similar to the proof of the above lemma.

The following theorem follows from the above two lemmas.

Theorem 15 \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIFC-ideals of \( X \) iff \( \bigoplus (A \times B) = (M_A \times M_B, M_A \times M_B) \) and \( \bigotimes (A \times B) = (N_A \times N_B, N_A \times N_B) \) are IVIFC-ideal of \( X \times X \).

Definition 21 Let \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) is IVIF BG-subalgebra of \( X \). For \([s_1, s_2],[t_1, t_2] \in D[0,1], \) the set \( U(M_A \times M_B : [s_1, s_2]) = \{(x,y) \in X \times X \mid (M_A \times M_B)(x,y) \geq [s_1, s_2]\} \) is called upper \([s_1, s_2]\)-level of \( A \times B \) and \( L(N_A \times N_B : [t_1, t_2]) = \{(x,y) \in X \times X \mid (N_A \times N_B)(x,y) \leq [t_1, t_2]\} \) is called lower \([t_1, t_2]\)-level of \( A \times B \).

Theorem 16 For any IVIFS \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \), \( A \times B \) is an IVIFC-ideals of \( X \times X \) iff the non-empty upper \([s_1, s_2]\)-level cut \( U(M_A \times M_B : [s_1, s_2]) \) and the non-empty lower \([t_1, t_2]\)-level cut \( L(N_A \times N_B : [t_1, t_2]) \) are closed ideals of \( X \times X \) for any \([s_1, s_2]\) and \([t_1, t_2]\) \in D[0,1].

Proof: Let \( A = (M_A, N_A) \) and \( B = (M_B, N_B) \) are IVIFC-ideals of \( X \), therefore for any \((x,y) \in X \times X \),
\[
(M_A \times M_B)((0,0)*(x,y)) \geq (M_A \times M_B)(x,y)
\]
and
\[
(N_A \times N_B)((0,0)*(x,y)) \leq (N_A \times N_B)(x,y).
\]

For \([s_1, s_2] \in D[0,1], \) if \( (M_A \times M_B)(x,y) \geq [s_1, s_2]. \)

That is, \( (M_A \times M_B)((0,0)*(x,y)) \geq [s_1, s_2]. \)

This implies, \((0,0)*(x,y) \in U(M_A \times M_B : [s_1, s_2]). \)

Let \((x,y),(x', y') \in X \times X \) such that \((x,y)*(x', y') \in U(M_A \times M_B : [s_1, s_2])\) and \((x', y') \in U(M_A \times M_B : [s_1, s_2]). \)

Now, \( (M_A \times M_B)(x,y) \geq \text{rmin}\{(M_A \times M_B)((x,y)*(x', y')),(M_A \times M_B)(x', y')\} \)
\[
\geq \text{rmin}\{[s_1, s_2],[s_1, s_2]\}
\]
This implies, \((x, y) \in U(M_A \times M_B : [s_1, s_2])\).

Thus \(U(M_A \times M_B : [s_1, s_2])\) is closed ideal of \(X \times X\).

Similarly, \(L(N_A \times N_B : [t_1, t_2])\) is closed ideal of \(X \times X\).

Conversely, let \((x, y) \in X \times X\) such that \((M_A \times M_B)(x, y) = [s_1, s_2]\) and \((N_A \times N_B)(x, y) = [t_1, t_2]\). This implies, \((x, y) \in U(M_A \times M_B : [s_1, s_2])\) and \((x, y) \in L(N_A \times N_B : [t_1, t_2])\). Since \((0, 0) \ast (x, y) \in U(M_A \times M_B : [s_1, s_2])\) and \((0, 0) \ast (x, y) \in L(N_A \times N_B : [t_1, t_2])\) (by definition of closed ideal). Therefore, \((M_A \times M_B)(0, 0) \ast (x, y) \geq [s_1, s_2]\) and \((N_A \times N_B)(0, 0) \ast (x, y) \leq [t_1, t_2]\). This gives, \((M_A \times M_B)((0, 0) \ast (x, y)) \geq (M_A \times M_B)(x, y)\) and \((N_A \times N_B)((0, 0) \ast (x, y)) \leq (N_A \times N_B)(x, y)\). Hence, \(A \times B\) is an IVIFC-ideal of \(X \times X\).

7. Conclusions and Future Work

In the present paper, we have introduced the concept of IVIF-ideal and IVIFC-ideal of BG-algebras are introduced and investigated some of their useful properties. The product of IVIF BG-subalgebra has been introduced and some important properties of it are also studied. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BF-algebras, lattices and Lie algebras.

It is our hope that this work would other foundations for further study of the theory of BG-algebras. The results obtained here probably be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, genetic algorithm, neural networks, expert systems, decision making, automata theory and medical diagnosis.

In our future study of fuzzy structure of BG-algebra, the following topics may be considered:

(i) To find interval-valued intuitionistic \((T,S)\)-fuzzy ideals, where \(S\) and \(T\) are given imaginable triangular norms;

(ii) To get more results in IVIFC-ideals of BG-algebra and their applications;

(iii) To find \((\varepsilon, \varepsilon \lor q)\)-interval-valued intuitionistic fuzzy ideals of BG-algebras.

8. REFERENCES

Authors

Tapan Senapati received his Bachelor of Science degree with honours in Mathematics in 2006 from Midnapore College, Pashim Medinipur, West Bengal, India and Master of Science degree in Mathematics in 2008 from Vidyasagar University, West Bengal, India. His research interest includes fuzzy sets, intuitionistic fuzzy sets, fuzzy algebra and lattice valued triangular norm.

Monoranjan Bhowmik received his M. Sc in Mathematics from Indian Institute of Technology, Kharagpur, West Bengal, India and Ph.D from Vidyasagar University, India in 1995 and 2008 respectively. He is a faculty member of V.T.T. College, Paschim Midnapore, West Bengal, India. His main scientific interest concentrates on discrete mathematics, fuzzy sets, intuitionistic fuzzy sets, fuzzy matrices, intuitionistic fuzzy matrices and fuzzy algebra.

Madhumangal Pal received his M. Sc from Vidyasagar University, India and Ph.D from Indian Institute of Technology, Kharagpur, India in 1990 and 1996 respectively. He is engaged in research since 1991. In 1996, he received Computer Division Award from Institution of Engineers (India), for best research work. During 1997 to 1999 he was a faculty member of Midnapore College and since 1999 he has been at the Vidyasagar University, India. His research interest includes computational graph theory, parallel algorithms, data structure, combinatorial algorithms, genetic algorithms, fuzzy sets, intuitionistic fuzzy sets, fuzzy matrices, intuitionistic fuzzy matrices, fuzzy game theory and fuzzy algebra.