Ranking Generalized Fuzzy Numbers using centroid of centroids

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Abstract

This paper describes a ranking method for ordering fuzzy numbers based on Area, Mode, Spreads and Weights of generalized fuzzy numbers. The area used in this method is obtained from the generalized trapezoidal fuzzy number, first by splitting the generalized trapezoidal fuzzy numbers into three triangles and then calculating the Centroids of each triangle followed by the centroid of these Centroids and then finding the area of this centroid from the original point. In this paper, we also apply mode and spreads in those cases where the discrimination is not possible. Some important results like linearity of ranking function and other properties are proved which are useful for proposed approach. This method is simple in evaluation and can rank various types of fuzzy numbers and also crisp numbers which are considered to be a special case of fuzzy numbers.

Keywords:
Ranking function; Centroid; Centroid points; Generalized trapezoidal fuzzy numbers

1. Introduction

Ranking fuzzy numbers plays a vital role in decision making. Most of the real problems that exist in natural world are fuzzy, than probabilistic or deterministic. In some cases the fuzzy numbers must be ranked before an action is taken by a decision maker. Since the inception of fuzzy sets by Zadeh [1] in 1965, many authors have proposed different methods for ranking fuzzy numbers. However, due to the complexity of the problem, there is no method which gives a satisfactory result to all situations. Most of the methods proposed so far are non-discriminating, counter-intuitive and some produce different rankings for the same situation and some methods cannot rank crisp numbers. Ranking fuzzy numbers was first proposed by Jain [2] for decision making in fuzzy situations by representing the ill-defined quantity as a fuzzy set. Since then, various procedures to rank fuzzy quantities are proposed by various researchers.

Yager [3], first used horizontal coordinate of the centroid point in ranking fuzzy numbers. Murakami et al. [4] have used both the horizontal and vertical coordinates of the centroid point as the ranking index. Cheng[5] proposed a distance index which is based on both horizontal and vertical coordinate of the centroid point as he pointed out in certain cases, the horizontal coordinates plays an important role than vertical coordinate of centroid point. This occurs when left and right spreads of fuzzy numbers are same. Chu and Tsao[6] proposed an area method to rank the fuzzy numbers by calculating area between centroid point and the original point.

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In this paper we compare not only the ranking fuzzy numbers using centroid but also ranking fuzzy numbers using area compensation by Fortemps and Roubens [13], Liou and Wang [14] presented ranking fuzzy numbers with integral value, Chen [15] presented ranking fuzzy numbers with maximizing set and minimizing set, an approach for ranking trapezoidal fuzzy numbers by Abbasbandy and Hajjari [16], fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads by Chen and Chen [17], and also the ranking proposed by Amit Kumar et al. [18] on ranking generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread.

In this paper a new method is proposed which is based on centroid of Centroids to rank fuzzy quantities. In a trapezoidal fuzzy number, first the trapezoid is split into three triangles and the centroids of these three triangles are calculated followed by the calculation of the centroid of these centroids. Finally, a ranking procedure is defined which is the area between the centroid of Centroids and the original point and also uses mode and spreads in those cases where the discrimination is not possible. In section 2, we briefly introduce fuzzy definitions and arithmetic operations. Section 3 presents the proposed new method. In Section 4, some important results like linearity of ranking function and other properties are proved which are useful for proposed approach. In Section 5, the proposed method has been explained with examples which describe the advantages and the efficiency of the method. In Section 6, the method demonstrates its power by comparing with other methods that exist in literature. Finally, the conclusions of the work are presented in Section 7.

2. Fuzzy concepts

In this section, some definitions and arithmetic operations on fuzzy set theory are reviewed.

**Definition 1.** Let \( U \) be a universe set. A fuzzy set \( \tilde{A} \) of \( U \) is defined by a membership function \( f_{\tilde{A}} : U \to [0,1] \), where \( f_{\tilde{A}}(x) \) is the degree of \( x \) in \( \tilde{A} \), \( \forall x \in U \).

**Definition 2.** A fuzzy set \( \tilde{A} \) of universe set \( U \) is normal if and only if \( \sup_{x \in U} f_{\tilde{A}}(x) = 1 \).

**Definition 3.** A fuzzy set \( \tilde{A} \) of universe set \( U \) is convex if and only if

\[
 f_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\left( f_{\tilde{A}}(x), f_{\tilde{A}}(y) \right), \forall x, y \in U \text{ and } \lambda \in [0,1].
\]

**Definition 4.** A fuzzy set \( \tilde{A} \) of universe set \( U \) is a fuzzy number iff \( \tilde{A} \) is normal and convex on \( U \).
Definition 5. A real fuzzy number \( \tilde{A} \) is described as any fuzzy subset of the real line \( \tilde{A} \) with membership function \( f_\tilde{A}(x) \) possessing the following properties:

1. \( f_\tilde{A}(x) \) is a continuous mapping from \( \mathbb{R} \) to the closed interval \( [0, w] \) where \( 0 < w \leq 1 \)

2. \( f_\tilde{A}(x) = 0 \), for all \( x \in (-\infty, a] \)

3. \( f_\tilde{A}(x) \) is strictly increasing on \( [a, b] \)

4. \( f_\tilde{A}(x) = 1 \), for all \( x \in [b, c] \)

5. \( f_\tilde{A}(x) \) is strictly decreasing on \( [c, d] \)

6. \( f_\tilde{A}(x) = 0 \), for all \( x \in (d, \infty] \), where \( a, b, c, d \) are real numbers

Definition 6. The membership function of the real fuzzy number \( \tilde{A} \) is given by

\[
 f_\tilde{A}(x) = \begin{cases} 
 f^L_\tilde{A}, & a \leq x \leq b, \\
 w, & b \leq x \leq c, \\
 f^R_\tilde{A}, & c \leq x \leq d, \\
 0, & \text{otherwise}, 
\end{cases}
\]  

(1) where \( 0 < w \leq 1 \)

is a constant, \( a, b, c, d \) are real numbers and \( f^L_\tilde{A} : [a, b] \to [0, w], f^R_\tilde{A} : [c, d] \to [0, w] \) are two strictly monotonic and continuous functions from \( \mathbb{R} \) to the closed interval \( [0, w] \). It is customary to write a fuzzy number as \( \tilde{A} = (a, b, c, d; w) \). If \( w = 1 \), then \( \tilde{A} = (a, b, c, d; 1) \) is a normalized fuzzy number, otherwise \( \tilde{A} \) is said to be a generalized or non-normal fuzzy number.

If the membership function \( f_\tilde{A}(x) \) is piecewise linear, then \( \tilde{A} \) is said to be a trapezoidal fuzzy number. The membership function of a trapezoidal fuzzy number is given by:

\[
 f_\tilde{A}(x) = \begin{cases} 
 \frac{w(x-a)}{b-a}, & a \leq x < b, \\
 w, & b \leq x \leq c, \\
 \frac{w(x-d)}{c-d}, & c \leq x < d, \\
 0, & \text{otherwise}, 
\end{cases}
\]  

(2)
If \( w=1 \), then \( \tilde{A} = (a,b,c,d;1) \) is a normalized trapezoidal fuzzy number and \( \tilde{A} \) is a generalized or non normal trapezoidal fuzzy number if \( 0 < w < 1 \).

The image of \( \tilde{A} = (a,b,c,d;w) \) is given by \( -\tilde{A} = (-d,-c,-b,-a;w) \).

As a particular case if \( b = c \), the trapezoidal fuzzy number reduces to a triangular fuzzy number given by \( \tilde{A} = (a,b,d;w) \). The value of ‘\( b \)’ corresponds with the mode or core and \([a, d]\) with the support. If \( w=1 \), then \( \tilde{A} = (a,b,d) \) is a normalized triangular fuzzy number \( \tilde{A} \) is a generalized or non normal triangular fuzzy number if \( 0 < w < 1 \).

**Definition 7** If \( \tilde{A} = (a_1,b_1,c_1,d_1;w_1) \) and \( \tilde{B} = (a_2,b_2,c_2,d_2;w_2) \) are two generalized trapezoidal fuzzy numbers, then

(i) \( \tilde{A} \oplus \tilde{B} = (a_1 + a_2,b_1 + b_2,c_1 + c_2,d_1 + d_2;\min(w_1,w_2)) \)

(ii) \( \tilde{A} \odot \tilde{B} = (a_1 - d_2,b_1 - c_2,c_1 - b_2,d_1 - a_2;\min(w_1,w_2)) \)

(iii) \( k \tilde{A} = (ka_1,kb_1,kc_1, kd_1;w_1); k > 0 \)

(iv) \( k \tilde{A} = (kd_1,kc_1, kb_1,k a_1;w_1); k < 0 \)

3. Proposed ranking Method

The Centroid of a trapezoid is considered as the balancing point of the trapezoid (Fig.1). Divide the trapezoid into three triangles. These three triangles are APC, QCD, and PQC respectively. Let the Centroids of the three triangles be \( G_1, G_2 \) & \( G_3 \) respectively. The centroid of Centroids \( G_1, G_2 \) & \( G_3 \) is taken as the point of reference to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of reference is that each Centroid point is a balancing point of each individual triangle, and the centroid of these Centroid points is a much more balancing point for a generalized trapezoidal fuzzy number. Therefore, this point would be a better reference point than the Centroid point of the trapezoid.
Consider a generalized trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d; w) \). (Fig.1). The Centroids of the three triangles are 

\[
G_1 = \left( \frac{a + b + c}{3}, \frac{w}{3} \right), \quad G_2 = \left( \frac{2c + d}{3}, \frac{w}{3} \right) \text{ and } G_3 = \left( \frac{b + 2c}{3}, \frac{2w}{3} \right)
\]

respectively. Equation of the line \( \overline{G_1 G_2} \) is \( y = \frac{w}{3} \) and \( G_3 \) does not lie on the line \( \overline{G_1 G_2} \). Therefore \( G_1, G_2 \) and \( G_3 \) are non-collinear and they form a triangle.

We define the centroid \( \overline{G_A(x_0, y_0)} \) of the triangle with vertices \( G_1, G_2 \) and \( G_3 \) of the generalized trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d; w) \) as

\[
G_A(x_0, y_0) = \left( \frac{a + 2b + 5c + d}{9}, \frac{4w}{9} \right)
\]

As a special case, for triangular fuzzy number \( \tilde{A} = (a, b, d; w) \). i.e., \( c = b \) the centroid of Centroids is given by

\[
G_A(x_0, y_0) = \left( \frac{a + 7b + d}{9}, \frac{4w}{9} \right)
\]

The ranking function of the generalized trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d; w) \). which maps the set of all fuzzy numbers to a set of real numbers is defined as:

\[
R(\tilde{A}) = x_0 \times y_0 = \frac{a + 2b + 5c + d}{9} \times \frac{4w}{9}
\]

This is the Area between the centroid of the Centroids \( \overline{G_A(x_0, y_0)} \) as defined in Eq.(3) and the original point.

The Mode \( (m) \) of the generalized trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d; w) \). is defined as:

\[
m = \frac{1}{2} \int_0^w (b + c)dx = \frac{w}{2}(b + c)
\]
The Spread \( s \) of the generalized trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d; w) \) is defined as:
\[
s = \int_0^w (d - a)dx = w(d - a)
\]  
(7)

The Left spread \( \text{ls} \) of the generalized trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d; w) \) is defined as:
\[
\text{ls} = \int_0^w (b - a)dx = w(b - a)
\]  
(8)

The Right spread \( \text{rs} \) of the generalized trapezoidal fuzzy number \( \tilde{A} = (a, b, c, d; w) \) is defined as:
\[
\text{rs} = \int_0^w (d - c)dx = w(d - c)
\]  
(9)

Using the above definitions we now define the ranking procedure of two generalized trapezoidal fuzzy numbers.

Let \( \tilde{A} = (a_1, b_1, c_1, d_1; w_1) \) and \( \tilde{B} = (a_2, b_2, c_2, d_2; w_2) \) be two generalized trapezoidal fuzzy numbers. The working procedure to compare \( \tilde{A} \) and \( \tilde{B} \) is as follows:

**Step 1:** Find \( R(\tilde{A}) \) and \( R(\tilde{B}) \)

- **Case (i)** If \( R(\tilde{A}) > R(\tilde{B}) \) then \( \tilde{A} > \tilde{B} \)
- **Case (ii)** If \( R(\tilde{A}) < R(\tilde{B}) \) then \( \tilde{A} < \tilde{B} \)
- **Case (iii)** If \( R(\tilde{A}) = R(\tilde{B}) \) comparison is not possible, then go to step 2.

**Step 2:** Find \( m(\tilde{A}) \) and \( m(\tilde{B}) \)

- **Case (i)** If \( m(\tilde{A}) > m(\tilde{B}) \) then \( \tilde{A} > \tilde{B} \)
- **Case (ii)** If \( m(\tilde{A}) < m(\tilde{B}) \) then \( \tilde{A} < \tilde{B} \)
- **Case (iii)** If \( m(\tilde{A}) = m(\tilde{B}) \) comparison is not possible, then go to step 3.

**Step 3:** Find \( s(\tilde{A}) \) and \( s(\tilde{B}) \)

- **Case (i)** If \( s(\tilde{A}) > s(\tilde{B}) \) then \( \tilde{A} < \tilde{B} \)
Case (ii) If $s(\tilde{A}) < s(\tilde{B})$ then $\tilde{A} > \tilde{B}$

Case (iii) If $s(\tilde{A}) = s(\tilde{B})$ comparison is not possible, then go to step 4.

Step 4: Find $ls(\tilde{A})$ and $ls(\tilde{B})$

Case (i) If $ls(\tilde{A}) > ls(\tilde{B})$ then $\tilde{A} > \tilde{B}$

Case (ii) If $ls(\tilde{A}) < ls(\tilde{B})$ then $\tilde{A} < \tilde{B}$

Case (iii) If $ls(\tilde{A}) = ls(\tilde{B})$ comparison is not possible, then go to step 5.

Step 5: Examine $w_1$ and $w_2$

Case (i) If $w_1 > w_2$ then $\tilde{A} > \tilde{B}$

Case (ii) If $w_1 < w_2$ then $\tilde{A} < \tilde{B}$

Case (iii) If $w_1 = w_2$ then $\tilde{A} = \tilde{B}$

In this section some important results which are the basis for defining the ranking procedure in section 3 are discussed and proved.

**Proposition 4.1**

The ranking function defined in section 3 by means of Eq. (3) is a linear function for normalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; 1)$, i.e., $R(\tilde{A}) = \frac{a + 2b + 5c + d}{9} \times \frac{4}{9}$.

If $\tilde{A} = (a_1, b_1, c_1, d_1; 1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; 1)$ are two normalized trapezoidal fuzzy numbers, then

(i) $R\left(k_1 \tilde{A} \oplus k_2 \tilde{B}\right) = k_1 R(\tilde{A}) \oplus k_2 R(\tilde{B}), k_1, k_2 \in \mathbb{R}$

(ii) $R\left(-\tilde{A}\right) = -R(\tilde{A})$

(iii) $R\left(\tilde{A} \oplus -\tilde{A}\right) = 0$

Proof (i):

case (i) Let $k_1, k_2 > 0$
\[ k_1 A \oplus k_2 B = (k_1 a_1 + k_2 a_2, k_1 b_1 + k_2 b_2, k_1 c_1 + k_2 c_2, k_1 d_1 + k_2 d_2) \]

\[
R\left( k_1 A \oplus k_2 B\right) = \left(\frac{k_1(a_1 + k_2 a_2) + 2(k_1 b_1 + k_2 b_2) + 5(k_1 c_1 + k_2 c_2) + (k_1 d_1 + k_2 d_2)}{9}\right)^\frac{4}{9} \\
= \left(\frac{k_1(a_1 + 2b_1 + 5c_1 + d_1) + k_2(a_2 + 2b_2 + 5c_2 + d_2)}{9}\right)^\frac{4}{9} \\
= k_1 R(A) \oplus k_2 R(B) 
\]

Similarly the result can be proved for case (ii) \( k_1 > 0, k_2 < 0 \) and case (iii). \( k_1 < 0, k_2 > 0 \)

**Proof (ii):**

Let \( A = (a, b, c, d) \Rightarrow -\tilde{A} = (-d, -c, -b, -a) \)

\[
R(-\tilde{A}) = \left(-\frac{d - 5c - 2b - a}{9}\right)^\frac{4}{9} \\
= -\left(\frac{a + 2b + 5c + d}{9}\right)^\frac{4}{9} \\
R(-\tilde{A}) = - R(\tilde{A})
\]

**Proof (iii):**

\[
R\left( [\tilde{A}] \oplus [\tilde{A}'] \right) = R\left( \tilde{A} \right) \oplus R\left( \tilde{A}' \right) \quad \text{(by (i))} \\
= R\left( \tilde{A} \right) \oplus R\left( \tilde{A} \right) \quad \text{(by (ii))} \\
= 0.
\]

**Proposition 4.2**

Let \( \tilde{A} = (a_1, b_1, c_1, d_1; w_1) \) and \( \tilde{B} = (a_2, b_2, c_2, d_2; w_2) \) are two generalized trapezoidal fuzzy numbers such that

\[
R\left( \tilde{A} \right) = R\left( \tilde{B} \right) \quad m\left( \tilde{A} \right) = m\left( \tilde{B} \right) \quad s\left( \tilde{A} \right) = s\left( \tilde{B} \right)
\]

(i) \( l_s(\tilde{A}) > l_s(\tilde{B}) \Leftrightarrow w_1 b_1 > w_2 b_2 \)

(ii) \( l_s(\tilde{A}) > l_s(\tilde{B}) \Leftrightarrow w_1 b_1 < w_2 b_2 \)

(iii) \( l_s(\tilde{A}) = l_s(\tilde{B}) \Leftrightarrow w_1 b_1 = w_2 b_2 \)

**Proof:** From the assumptions

\[
R\left( \tilde{A} \right) = R\left( \tilde{B} \right) \Rightarrow w_1 (a_1 + 2b_1 + 5c_1 + d_1) = w_2 (a_2 + 2b_2 + 5c_2 + d_2) \quad (10)
\]
\[
m\left(\tilde{A}\right) = m\left(\tilde{B}\right) \implies w_1(b_1 + c_1) = w_2(b_2 + c_2) \quad (11)
\]
\[
s\left(\tilde{A}\right) = s\left(\tilde{B}\right) \implies w_1(d_1 - a_1) = w_2(d_2 - a_2) \quad (12)
\]
Solving (10), (11) and (12) we get
\[
w_1a_1 = w_2a_2
\]
\[
w_1d_1 = w_2d_2
\]
\[
w_1c_1 = w_2c_2
\]
Now to prove (i):
\[
\begin{align*}
ls\left(\tilde{A}\right) &> ls\left(\tilde{B}\right) \\
\iff w_1(b_1 - a_1) &> w_2(b_2 - a_2) \\
\iff w_1b_1 &> w_2b_2 \quad (\because w_1a_1 = w_2a_2)
\end{align*}
\]
Now to prove (ii):
\[
\begin{align*}
ls\left(\tilde{A}\right) &< ls\left(\tilde{B}\right) \\
\iff w_1(b_1 - a_1) &< w_2(b_2 - a_2) \\
\iff w_1b_1 &< w_2b_2 \quad (\because w_1a_1 = w_2a_2)
\end{align*}
\]
Now to prove (iii):
\[
\begin{align*}
ls\left(\tilde{A}\right) &= ls\left(\tilde{B}\right) \\
\iff w_1(b_1 - a_1) &= w_2(b_2 - a_2) \\
\iff w_1b_1 &= w_2b_2 \quad (\because w_1a_1 = w_2a_2)
\end{align*}
\]
**Corollary**: All the results of proposition 4.2 also hold for right spread.

**Proposition 4.3**

Let \( \tilde{A} = (a_1, b_1, c_1, d_1; w_1) \) and \( \tilde{B} = (a_2, b_2, c_2, d_2; w_2) \) are two generalized trapezoidal fuzzy numbers such that

\[
R\left(\tilde{A}\right) = R\left(\tilde{B}\right) ; \quad m\left(\tilde{A}\right) = m\left(\tilde{B}\right) ; \quad s\left(\tilde{A}\right) = s\left(\tilde{B}\right) \quad \text{then}
\]
\[
(i) \quad ls\left(\tilde{A}\right) > ls\left(\tilde{B}\right) \iff rs\left(\tilde{A}\right) > rs\left(\tilde{B}\right)
\]
\[
(ii) \quad ls\left(\tilde{A}\right) < ls\left(\tilde{B}\right) \iff rs\left(\tilde{A}\right) < rs\left(\tilde{B}\right)
\]
\[
(iii) \quad ls\left(\tilde{A}\right) = ls\left(\tilde{B}\right) \iff rs\left(\tilde{A}\right) = rs\left(\tilde{B}\right)
\]

**Proof:**

From proposition 4.2, for the above assumptions we have
\[ w_1(b_1 + c_1) = w_2(b_2 + c_2) \]
\[ w_1a_1 = w_2a_2 \]
\[ w_1d_1 = w_2d_2 \]
\[ w_1c_1 = w_2c_2 \]

Now to prove (i):
\[
ls\left( \tilde{A} \right) > ls\left( \tilde{B} \right) \\
\iff w_1b_1 - w_2b_2 > 0 \quad (\therefore w_1c_1 = w_2c_2) \quad \text{(from proposition 4.2)}
\]
\[
\iff w_2c_2 - w_1c_1 > 0 \quad (\therefore w_1d_1 = w_2d_2) \quad \text{(from proposition 4.2)}
\]
\[
\iff -w_1c_1 > -w_2c_2
\]
\[
\iff w_1(d_1 - c_1) > w_2(d_2 - c_2) \quad (\therefore w_1d_1 = w_2d_2)
\]
\[
\iff rs\left( \tilde{A} \right) > rs\left( \tilde{B} \right)
\]

Similarly (ii) and (iii) can be proved.

5. Numerical Examples

In this section, the proposed method is first explained by ranking some fuzzy numbers.

Example 5.1
Let \( \tilde{A} = (3,5,7;1) \) and \( \tilde{B} = (4,5,\frac{51}{8};1) \)

Then \( G_{\tilde{A}}\left(\bar{x_0}, \bar{y}_0\right) = (5,0.4444) \) and \( G_{\tilde{B}}\left(\bar{x_0}, \bar{y}_0\right) = (5.0416, 0.4444) \)

Therefore, \( R(\tilde{A}) = 2.222 \) and \( R(\tilde{B}) = 2.2405 \)
Since \( R(\tilde{A}) < R(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B} \)

Example 5.2
Let \( \tilde{A} = (0,1,2;1) \) and \( \tilde{B} = \left(\frac{1}{5}, 1, \frac{7}{4};1\right) \)

Then \( G_{\tilde{A}}\left(\bar{x_0}, \bar{y}_0\right) = (1,0.4444) \) and \( G_{\tilde{B}}\left(\bar{x_0}, \bar{y}_0\right) = (0.9944, 0.4444) \)

Therefore, \( R(\tilde{A}) = 0.4444 \) and \( R(\tilde{B}) = 0.4419 \)
Since \( R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B} \)

Example 5.3
Let \( \tilde{A} = (0,1,2;1) \Rightarrow -\tilde{A} = (-2,-1,0;1) \) and \( \tilde{B} = (\frac{1}{5}, 1, \frac{7}{4};1) \Rightarrow -\tilde{B} = \left(-\frac{7}{4}, -1, -\frac{1}{5};1\right) \)

\( G_{\tilde{A}}\left(\bar{x_0}, \bar{y}_0\right) = (-1,0.4444) \) and \( G_{\tilde{B}}\left(\bar{x_0}, \bar{y}_0\right) = (-0.9944, 0.4444) \)
Therefore, $R(-\tilde{A}) = -0.4444$ and $R(-\tilde{B}) = -0.44191$

Since $R(-\tilde{A}) < R(-\tilde{B}) \Rightarrow -\tilde{A} < -\tilde{B}$

From examples 5.2 and 5.3 we see that the proposed method can rank fuzzy numbers and their images as it is proved that $\tilde{A} > \tilde{B} \Rightarrow -\tilde{A} < -\tilde{B}$.

**Example 5.4**

Let $\tilde{A} = (0.1,0.3,0.5;1)$ and $\tilde{B} = (0.2,0.3,0.4;1)$

**Step 1:**

Then $G_A(x_o, y_o) = (0.3,0.4444)$ and $G_B(x_o, y_o) = (0.3,0.4444)$

Therefore, $R(\tilde{A}) = 0.1333$ and $R(\tilde{B}) = 0.1333$

Since $R(\tilde{A}) = R(\tilde{B})$

So go to step 2.

**Step 2:**

$m(\tilde{A}) = 0.3$ and $m(\tilde{B}) = 0.3$

Since $m(\tilde{A}) = m(\tilde{B})$, So go to step 3.

**Step 3:**

$s(\tilde{A}) = 0.4$ and $s(\tilde{B}) = 0.2$

Since $s(\tilde{A}) > s(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}$

**Example 5.5**

Let $\tilde{A} = (0.1,0.3,0.5;0.8)$ and $\tilde{B} = (0.1,0.3,0.5;1)$

Then $G_A(x_o, y_o) = (0.3,0.3555)$ and $G_B(x_o, y_o) = (0.3,0.4444)$

Therefore, $R(\tilde{A}) = 0.1066$ and $R(\tilde{B}) = 0.1333$

Since $R(\tilde{A}) < R(\tilde{B}) \Rightarrow \tilde{A} < \tilde{B}$

From example 5.5 it is clear that the proposed method can rank fuzzy numbers with different height and same spreads.

**Example 5.6**

Let $\tilde{A} = (0.1,0.2,0.4,0.5;1)$ and $\tilde{B} = (0.1,0.3,0.5;1)$

Then $G_A(x_o, y_o) = (0.3333,0.4444)$ and $G_B(x_o, y_o) = (0.3,0.4444)$
Therefore, \( R(\tilde{A}) = 0.1481 \) and \( R(\tilde{B}) = 0.13 \)

Since \( R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B} \)

6. Results and discussion

In this section the advantages of the proposed method is shown by comparing with other existing methods in literature, where the methods cannot discriminate fuzzy numbers. The results are shown in Table I and Table II.

Example 6.1

Consider two fuzzy numbers \( \tilde{A} = (1,4,5) \) and \( \tilde{B} = (2,3,6) \)

By Liou and Wang Method [14], it is clear that the two fuzzy numbers are equal for all the decision makers as

\[
I^L(\tilde{A}) = 4.5\alpha + (1 - \alpha)2.5 \quad \text{and} \quad I^L(\tilde{B}) = 4.5\alpha + (1 - \alpha)2.5
\]

Which is not even true by intuition.

By using our method we have

\[
G_A(\bar{x}_0, \bar{y}_0) = (3.77, 0.4444) \quad \text{and} \quad G_B(\bar{x}_0, \bar{y}_0) = (3.2222, 0.4444)
\]

Therefore, \( R(\tilde{A}) = 1.6788 \) and \( R(\tilde{B}) = 1.4319 \Rightarrow \tilde{A} > \tilde{B} \)

Since \( R(\tilde{A}) > R(\tilde{B}) \Rightarrow \tilde{A} > \tilde{B} \)

Example 6.2

Let \( \tilde{A} = (0.1, 0.3, 0.5; 1) \), \( \tilde{B} = (1, 1, 1; 1) \)

Cheng [5] ranked fuzzy numbers with the distance method using the Euclidean distance between the Centroid point and original point. Where as Chu and Tsao [6] proposed a ranking function which is the area between the centroid point and original point. Their centroid formulae are given by

\[
(x_0, y_0) = \left( \frac{w(d^2 - 2c^2 + 2b^2 - a^2 + dc - ab) + 3(c^2 - b^2)}{3w(d - c + b + a) + 6(c - b)}, \frac{w(1 + \frac{(b + c) - (a + d)(1 - w)}{(b + c - a - d) + 2(a + d)w})}{(b + c)} \right)
\]

\[
(x_0, y_0) = \left( \frac{w(d^2 - 2c^2 + 2b^2 - a^2 + dc - ab) + 3(c^2 - b^2)}{3w(d - c + b + a) + 6(c - b)}, \frac{w(1 + \frac{(b + c)}{a + b + c + d})}{(b + c)} \right)
\]

Both these centroid formulae cannot rank crisp numbers which are a special case of fuzzy numbers as it can be seen from the above formulae that the denominator in the first coordinate of their centroid formulae is zero, and hence centroid of crisp numbers are undefined for their formulae. By using our method, we have
\[ G_A(x_0, y_0) = (0.3, 0.4444) \text{ and } G_B(x_0, y_0) = (1.0, 4.444) \]

Therefore, \( R(A) = 0.1333 \text{ and } R(B) = 0.4444 \)

Since \( R(A) < R(B) \Rightarrow A < B \)

From this example it is proved that the proposed method can rank crisp numbers whereas, other methods failed to do so.

**Example 6.3**

Consider four fuzzy numbers

\[ \tilde{A}_1 = (0.1, 0.2, 0.3; 1), \tilde{A}_2 = (0.2, 0.3, 5.0.8; 1), \tilde{A}_3 = (0.3, 0.4, 0.9; 1), \tilde{A}_4 = (0.6, 0.7, 0.8; 1) \]

Which were ranked earlier by Yager[3], Fortemps and Roubens[13], Liou and Wang[14], and Chen and Lu [15] as shown in Table I.
It can be seen from Table I that none of the methods discriminates fuzzy numbers.

Yager [3] and Fortemps and Roubens [13] methods failed to discriminate the fuzzy numbers \( \tilde{A}_2 \) and \( \tilde{A}_1 \), Whereas the methods of Liou and Wang [14], and Chen and Lu [15] cannot discriminate the fuzzy numbers \( \tilde{A}_2 \), \( \tilde{A}_3 \), \( \tilde{A}_4 \), \( \tilde{A}_5 \), \( \tilde{A}_6 \), \( \tilde{A}_7 \).

By using our method, we have

\[
G_{\tilde{A}_1} (\tilde{x}_0, \tilde{y}_0) = (0.20, 0.4444), \quad G_{\tilde{A}_2} (\tilde{x}_0, \tilde{y}_0) = (0.50, 0.4444), \quad G_{\tilde{A}_3} (\tilde{x}_0, \tilde{y}_0) = (0.4444, 0.4444), \quad G_{\tilde{A}_4} (\tilde{x}_0, \tilde{y}_0) = (0.70, 0.4444)
\]

Therefore,

\[
R(\tilde{A}_1) = 0.0888, \quad R(\tilde{A}_2) = 0.2222, \quad R(\tilde{A}_3) = 0.1974, \quad R(\tilde{A}_4) = 0.3110 \quad \Rightarrow \quad \tilde{A}_1 > \tilde{A}_2 > \tilde{A}_3 > \tilde{A}_4
\]

**Example 6.4**

In this, we consider seven sets of fuzzy numbers available in literature and the comparative study is presented in Table II.

Set 1: \( \tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35) \) and \( \tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7) \)
Set 2: \( \tilde{A} = (0.1, 0.2, 0.4, 0.5; 1) \) and \( \tilde{B} = (0.1, 0.3, 0.3, 0.5; 1) \)
Set 3: \( \tilde{A} = (0.1, 0.2, 0.4, 0.5; 1) \) and \( \tilde{B} = (1, 1, 1, 1; 1) \)
Set 4: $\tilde{A} = (-0.5, -0.3, -0.3, -0.1; 1)$, and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$
Set 5: $\tilde{A} = (0.3, 0.5, 0.5; 1)$, and $\tilde{B} = (0.1, 0.6, 0.6, 0.8; 1)$
Set 6: $\tilde{A} = (0.0, 4.0, 6.0, 8.1)$, $\tilde{B} = (0.2, 0.5, 0.5, 0.9; 1)$ and $\tilde{C} = (0.1, 0.6, 0.7, 0.8; 1)$,
Set 7: $\tilde{A} = (0.1, 0.2, 0.4, 0.5; 1)$, and $\tilde{B} = (-2.0, 0.2; 1)$

Table II A Comparison of the ranking results for different approaches

<table>
<thead>
<tr>
<th>Methods</th>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
<th>Set 5</th>
<th>Set 6</th>
<th>Set 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cheng[5]</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} \approx \tilde{B}$</td>
<td>Not Comparable</td>
<td>$\tilde{A} \approx \tilde{B}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B} &lt; \tilde{C}$</td>
<td>Not Comparable</td>
</tr>
<tr>
<td>Chui and Tsaao[6]</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} \approx \tilde{B}$</td>
<td>Not Comparable</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B} &lt; \tilde{C}$</td>
<td>Not Comparable</td>
</tr>
<tr>
<td>Chen and Chen (2007)[11]</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>Not Comparable</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B} &lt; \tilde{C}$</td>
<td>Not Comparable</td>
</tr>
<tr>
<td>Abbasbandy and Hajjari (2009)[16]</td>
<td>$\tilde{A} \approx \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} \approx \tilde{B}$</td>
<td>$\tilde{A} \approx \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B} &lt; \tilde{C}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td></td>
</tr>
<tr>
<td>Chen and Chen (2008)[17]</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B} &lt; \tilde{C}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td></td>
</tr>
<tr>
<td>Kumar et al.[18]</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B} &lt; \tilde{C}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td></td>
</tr>
<tr>
<td>Proposed method</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B}$</td>
<td>$\tilde{A} &lt; \tilde{B} &lt; \tilde{C}$</td>
<td>$\tilde{A} &gt; \tilde{B}$</td>
<td></td>
</tr>
</tbody>
</table>

7. Conclusions and future work

This paper proposes a method that ranks fuzzy numbers which is simple and concrete. This method ranks trapezoidal as well as triangular fuzzy numbers and their images. This method also ranks crisp numbers which are special case of fuzzy numbers whereas some methods proposed in literature cannot rank crisp numbers. This method which is simple and easier in calculation not only gives satisfactory results to well defined problems, but also gives a correct ranking order to problems. Comparative examples are used to illustrate the advantages of the proposed method. Application of this ranking procedure in various decision making problems such as, fuzzy risk analysis and in fuzzy optimization like network analysis is left as future work.

References