Lattices of Fuzzy Measures Defined on Hilbert Spaces

Manju Cherian and K. Sudheer
Associate Professor, Department of Mathematics,
Farook College, Kozhikode, Kerala-673 632
India.
sudheer@farookcollege.ac.in

Abstract

A new type of translation invariant and lower semi continuous fuzzy measure, called VGFM (Vector Generated Fuzzy Measure), on the class of subsets of a real Hilbert space is introduced. It measures a subset of the Hilbert space as a projection of the set along a given unit vector in the Hilbert space. Any VGFM uniquely defines two orthogonal subspaces called the support space and null space whose direct sum is the Hilbert space. An equivalence of VGFM is defined related to the support space. A partial order relation similar to that of absolute continuity is defined on the class of all VGFM. It is proved that this partial order makes the class of all VGFM a lattice. Further properties of the lattice of VGFM are studied.

Key words:
Vector Generated Fuzzy Measure, Support space, Null space, Orthogonal VGFM, Absolute Continuity, Lattice, Join, Meet, Atomic VGFM, Finite Dimensional VGFM.

1. Introduction

A new type of fuzzy measure on a real Hilbert space was defined [6] by Manju Cherian and K. Sudheer. It was proved [7] that the fuzzy measure of a compact and convex set resembles the concept of length of an interval as the difference between the end points. An equivalence relation based on the support space of the VGFM is defined. On the class of all VGFM on a Hilbert space, a partial order relation is introduced. This partial order is compared with the absolute continuity of classical measures. It is proved that the partial order makes the class of VGFM a lattice.
2. Preliminaries

Definition 2.1 (Fuzzy Measure). Let $X$ be a non-empty set and $C$, a class of subsets of $X$. A fuzzy measure is an extended real valued function $\mu : C \rightarrow [0, \infty]$ satisfying the conditions

$FM_1 : \mu(\Phi) = 0$ whenever $\Phi \in C$ and

$FM_2 :$ For $A, B \in C$ with $A \subseteq B$, the condition $\mu(A) < \mu(B)$ holds.

The lower and upper semi-continuity are respectively defined as follows.

$FM_3 :$ (Lower semi-continuity) If $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots \in C$ and if $E = \bigcup_{n=1}^{\infty} E_n \in C$ then $\lim_{n \to \infty} \mu(E_n) = \mu(E)$;

$FM_4 :$ (Upper semi-continuity) If $E_1, E_2, \ldots, E_n, \ldots \in C$ are such that $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_n \supseteq \cdots$, $\mu(E_k) < \infty$ for some $k$ and if $E = \bigcap_{n=1}^{\infty} E_n \in C$ then $\lim_{n \to \infty} \mu(E_n) = \mu(E)$.

Definition 2.2 (Null Additivity). Let $X$ be a non-empty set and let $C$ be a class of subsets of $X$. A set function $\mu : C \rightarrow [-\infty, \infty]$ is said to be Null-Additive if $\mu(E \cup F) = \mu(E)$ whenever $E, F \in C$, $E \cap F = \Phi$ and $\mu(F) = 0$.

Definition 2.3 (Vector Generated Fuzzy Measure (VГFM)). Let $H$ be a real Hilbert space and let $x \in H$ be a unit vector.

Define a function $\mu_x : \mathcal{P}(H) \rightarrow \mathbb{R}^+ \cup \{0, \infty\}$ as

$$\mu_x(A) = \sup\{\langle a - b, x \rangle : a, b \in A\}.$$

Then $\mu_x$ is called a a VГFM.

The VГFM $\mu_x$ possesses the following properties, ([6], [7]).

1. It is translation invariant fuzzy measure defined on the power set of the Hilbert space;

2. There exist two closed subspaces $V$ and $W$ of $H$ respectively called the support space and null space of $\mu_x$ satisfying

   (i) $x \in V$ and $x \perp W$,

   (ii) $V \perp W$,

   (iii) $\mu_x(B) = 0$ for all $B \subseteq W$.

   (iv) If $\mu_x(A) \neq 0$ for $A \subseteq H$, then $\exists A_V \subseteq V$ such that $\mu_x(A) = \mu_x(A_V)$.  


3. The VGFMs of a non-empty subset of the support space is zero if and only if the set is a singleton;

4. VGFMs is a lower semi-continuous fuzzy measure;

5. VGFMs does not satisfy the upper semi-continuity and null-additivity properties;

6. If for $A \subseteq H$, $\mu_x(A) < \infty$, then there exists a vector $h \in H$ such that $\mu_x(A) = \langle h, x \rangle$. In case there are two such elements $h, h' \in H$ satisfying $\langle h, x \rangle = \mu_x(A) = \langle h', x \rangle$, then $h - h' \in W$;

7. If $\exists h \in A$ with $\mu_x(A) = \langle h, x \rangle$, then $0 \leq \langle a, x \rangle \leq 2\langle h, x \rangle \forall a \in A$;

8. If $A$ is a closed convex subset of $H$, then there exists a pair of vectors $h, k \in A$ satisfying $\mu_x(A) = |\langle h - k, x \rangle|$. So the VGFMs of a compact convex subset of a Hilbert space satisfies a condition similar to the length of a closed interval in $R$ as a difference between its end points.

### 3. Lattice of VGFMs

Precedence is introduced among the class of all VGFMs. It is proved that this resembles the absolute continuity of fuzzy measures. Moreover it forms a partial order making the set of all VGFMs a lattice. Precedence is given the same notation as that of absolute continuity of fuzzy measures.

**Definition 3.1.** Let $\mu$ and $\nu$ be two VGFMs on a Hilbert space $H$. Define $\mu \cong \nu$ if the support spaces of $\mu$ and $\nu$ are equal and the null spaces of $\mu$ and $\nu$ are equal.

**Theorem 3.2.** The relation $\mu \cong \nu$ defined for two VGFMs $\mu$ and $\nu$ is an equivalence relation.

Proof. The conditions of reflexivity, symmetry and transitivity follow directly from the definition.

**Note 3.3.** Under the equivalence defined, the collection of all VGFMs becomes an equivalence class. Each member of this class represents a collection of VGFMs equivalent to it. If the Hilbert space is assumed to be separable, it is possible to have a VGFMs corresponding to every orthogonal pair of subspaces making a decomposition of the Hilbert.
Theorem 3.4. For each pair of non-empty subspaces of a Hilbert space which are orthogonal complements of each other, there is a VGFM for which these two subspaces chosen are respectively the support and null spaces.

Proof. Let $H$ be a separable Hilbert space. Let $V$ and $W$ be two non-empty subspaces of $H$ which are orthogonal complements of each other. Then $V \oplus W = H$. Let $\mathcal{B} = \{e_1, e_2, e_3, \cdots, e_n, \cdots\}$ be a complete orthonormal basis of $H$. The set $\mathcal{I} = \{k : k \in \mathbb{N}, e_k \perp W\}$ is a non-empty set of natural numbers. By the choice of $\mathcal{I}$, $e_k \in V \forall k \in \mathcal{I}$. Let $\{\alpha_k : k \in \mathcal{I}\}$ be a square summable sequence of non-zero real numbers such that

$$\sum_{k=1}^{\infty} \alpha_k = 1.$$

Let $x = \sum_{k=1}^{\infty} \alpha_k \cdot e_k$. Then $x$ is a unit vector in $V$. By the choice of the set $\mathcal{I}$, it follows that for any set $B \subseteq W$, $\mu_x(B) = 0$ and the condition that the terms of the sequence $\{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_k, \cdots\}$ are non-zero guarantee that the support space of $\mu_x$ is $V$ and the null space is $W$.

Note 3.5. The fuzzy measure corresponding to a given pair of closed subspaces which are orthogonal complements to one another is unique in the sense of 3.1.

Definition 3.6. Let $\mu$ and $\nu$ be two VGFM$s$ on a Hilbert space $H$. We say that $\mu$ precedes $\nu$, denoted as $\mu << \nu$, if the support space of $\mu$ is contained in the support space of $\nu$ and so the null space of $\nu$ is contained in the null space of $\mu$.

Proposition 3.7. Let $\mu$ and $\nu$ be two VGFM$s$ on a Hilbert space $H$ such that $\mu << \nu$. Then for any $A \subseteq H$, $\nu(A) = 0 \Rightarrow \mu(A) = 0$. In other words, the precedence defined is a generalisation of the concept of absolute continuity of classical measures.
Proof. Let \( \mu \) and \( \nu \) be two VGFM\s on a Hilbert space \( H \). Then there exists unit vectors \( x \) and \( y \) such that \( \mu = \mu_x \) and \( \nu = \nu_y \). Let \( V_x \) and \( W_x \) respectively be the support space and null space of \( \mu_x \). Similarly let \( V_y \) and \( W_y \) respectively be the support space and null space of \( \nu_y \). Since \( \mu_x \ll \nu_y \), \( V_x \subseteq V_y \) and so \( W_y \subseteq W_x \). Now for a set \( A \subseteq H \) the condition

\[
\nu(A) = 0 \Rightarrow \nu_y(A) = 0 \\
\Rightarrow A = \emptyset \text{ or } A \subseteq W_y \\
\Rightarrow A = \emptyset \text{ or } A \subseteq W_x \\
\Rightarrow \mu_x(A) = 0 \\
\Rightarrow \mu(A) = 0
\]

Thus \( \mu \ll \nu \) in the classical sense. \( \Box \)

Note 3.8. The precedence defined is now called the absolute continuity of VGFM\s. This justifies the use of the notation \( \mu \ll \nu \) for VGFM\s.

Theorem 3.9. The absolute continuity of VGFM\s defines a partial order on the class of all VGFM\s.

Proof. For any VGFM \( \alpha \) on \( H \), the support space of \( \alpha \) is a subspace of itself. So \( \alpha \ll \alpha \) and hence the relation is reflexive. Let \( \alpha \ll \beta \) and \( \beta \ll \alpha \). Then the support spaces of \( \alpha \) and \( \beta \) are subspaces of each other and so they become equal. In this case, by definition 3.1, \( \alpha \cong \beta \). So the relation is antisymmetric. Now let \( \alpha \ll \beta \) and \( \beta \ll \gamma \). Then the support space of \( \alpha \) is a subset of the support space of \( \beta \) and the support space of \( \beta \) is a subset of the support space of \( \gamma \). So the support space of \( \alpha \) is a subset of the support space of \( \gamma \). Thus the relation is transitive. Hence the relation is a partial order. \( \Box \)
Theorem 3.10. Let \( \mu_x \) and \( \mu_y \) be two VGFM's on a Hilbert space \( H \) where \( x \) and \( y \) are unit vectors in \( H \). Let \( V_x \) and \( W_x \) be respectively the support space and null space of \( \mu_x \), and let \( V_y \) and \( W_y \) be respectively the support space and null space of \( \mu_y \). Then \( V_{x \wedge y} = V_x \cap V_y \) and \( W_{x \wedge y} = W_x \oplus W_y \) are two subspaces of \( H \) satisfying \( H = V_{x \wedge y} \oplus W_{x \wedge y} \). Similarly \( V_{x \vee y} = V_x \oplus V_y \) and \( W_{x \vee y} = W_x \cap W_y \) are two subspaces of \( H \) satisfying \( H = V_{x \vee y} \oplus W_{x \vee y} \).

Proof. By definition of support and null spaces of the VGFM \( \mu_x \), \( V_x \) and \( W_x \) are orthogonal complements of each other. Similarly \( V_y \) and \( W_y \) are orthogonal complements of each other. Now \( V_{x \wedge y} = V_x \cap V_y \) being the intersection of two closed subspaces is a closed subspace of \( H \).

Let \( W_{x \wedge y} = V_{x \wedge y}^\perp \). Then

\[
\begin{align*}
z \in W_x &\Rightarrow \langle z, h \rangle = 0; \ \forall \ h \in V_x \\
&\Rightarrow \langle z, h \rangle = 0; \ \forall \ h \in V_{x \wedge y} \\
&\Rightarrow z \in W_{x \wedge y},
\end{align*}
\]

Hence \( W_x \subseteq W_{x \wedge y} \). Similarly \( W_y \subseteq W_{x \wedge y} \). Thus \( W_x \oplus W_y \subseteq W_{x \wedge y} \).

Now \( W_{x \wedge y} \) will be the smallest closed subspace of \( H \) containing both \( W_x \) and \( W_y \) since it is the orthogonal complement of the closed subspace \( V_{x \wedge y} \). Thus \( W_x \oplus W_y = W_{x \wedge y} \). By definition of orthogonal complements, the two closed subspaces \( V_{x \wedge y} \) and \( W_{x \wedge y} \) satisfy \( H = V_{x \wedge y} \oplus W_{x \wedge y} \).

To prove the second part, let \( W_{x \vee y} = W_x \cap W_y \) and \( V_{x \vee y} = W_{x \vee y}^\perp \). These are two closed subspaces of \( H \). Now
\[ h \in V_x \Rightarrow \langle h, z \rangle = 0; \ \forall \ z \in W_x \]
\[ \Rightarrow \langle h, z \rangle = 0; \ \forall \ z \in W_{x \land y} \]
\[ \Rightarrow h \in V_{x \lor y} \]

Hence \( V_x \subseteq V_{x \lor y} \). Similarly \( V_y \subseteq V_{x \lor y} \). Thus \( V_x \oplus V_y \subseteq V_{x \lor y} \).

Now \( V_{x \lor y} \) will be the smallest closed subspace of \( H \) containing both \( V_x \) and \( V_y \) since it is the orthogonal complement of the closed subspace \( W_{x \lor y} \). Thus \( V_x \oplus V_y = V_{x \lor y} \). The two closed subspaces \( V_{x \lor y} \) and \( V_{x \lor y} \cap V_{x \lor y} \) being the orthogonal complements of each other, satisfy \( H = V_{x \lor y} \oplus W_{x \lor y} \).

**Definition 3.11.** Let \( \mu_x \) and \( \mu_y \) be two VGFMs on a Hilbert space \( H \) where \( x \) and \( y \) are unit vectors in \( H \). Let \( V_x \) and \( W_x \) be respectively the support space and null space of \( \mu_x \). Similarly let \( V_y \) and \( W_y \) be respectively the support space and null space of \( \mu_y \).

**The Meet of \( \mu_x \) and \( \mu_y \)**: It is a VGF having \( V_x \cap V_y \) as the support space. It is denoted by \( \mu_{x \land y} \).

**The Join of \( \mu_x \) and \( \mu_y \)**: It is a VGF having \( W_x \cap W_y \) as the null space. It is denoted by \( \mu_{x \lor y} \).

**Theorem 3.12.** Under the partial order ‘\(<\<\)’, the set of all VGFMs on a Hilbert space form a lattice.

**Proof.** Since ‘\(<\<\)’ is a partial order among the class of all VGFMs, for any two VGFMs \( \mu_x \) and \( \mu_y \), the join \( \mu_{x \lor y} \) and the meet \( \mu_{x \land y} \) are VGFMs. These are defined uniquely up to the equivalence of VGFMs.
Note 3.13. 1. If $\mu_x \ll \mu_y$ are VGFMs, then the obvious properties of lattices namely $\mu_{x \wedge y}$ is the minimum of $\mu_x$ and $\mu_y$ and $\mu_{x \vee y}$ is the maximum of $\mu_x$ and $\mu_y$ are satisfied.

2. For separable Hilbert spaces, considering the complete orthonormal basis, further classification of VGFMs can be had.

Definition 3.14. A VGF is said to be atomic if its support space is one dimensional.

Definition 3.15. A VGF is said to be finite dimensional if its support space is of dimension $n$ for some natural number $n$.

Definition 3.16. Two VGFMs are said to be orthogonal if their support spaces are orthogonal.

Note 3.17. Let $\mathcal{B} = \{e_1, e_2, e_3, \ldots, e_n, \ldots\}$ be a complete orthonormal basis of a separable Hilbert space $H$. Then the collection $\\{\mu_{e_1}, \mu_{e_2}, \ldots, \mu_{e_n}, \ldots\}$ is an orthogonal family of atomic VGFMs.

Proposition 3.18. For any non-zero VGF $\mu$ defined on a separable Hilbert space $H$, there exists an atomic VGF $\nu$ such that $\nu \ll \mu$.

Proof. Let $\mathcal{B} = \{e_1, e_2, e_3, \ldots, e_n, \ldots\}$ be a complete orthonormal basis of $H$. Let $V$ be the support space and $W$ be the null space of the VGF $\mu$. Since $V$ is non-zero, there exists $h \in V$ such that $h \neq 0$. Then there will be $e_n \in \mathcal{B}$ such that $\langle h, e_n \rangle \neq 0$. Taking $\nu$ as the VGF corresponding to the unit vector $e_n$, the proposition follows. 

Proposition 3.19. Any finite dimensional VGF defined on a separable Hilbert space is the join of atoms.
Proof. Let $\mu$ be a finite dimensional VGFM on a separable Hilbert space $H$ with $V$ as the support space. Let $V$ be a $k$-dimensional subspace of $H$. Let $e_{n_1}, e_{n_2}, \ldots, e_{n_k}$ be the elements of the complete orthonormal basis of $H$ that span $V$. Clearly the VGFM$s$ $\mu_{e_{n_1}}, \mu_{e_{n_2}} \ldots$ and $\mu_{e_{n_k}}$ are atomic. Then $\mu_{e_{n_1}} << \mu$, $\mu_{e_{n_2}} << \mu \ldots$ and $\mu_{e_{n_k}} << \mu$. The VGFM defined as $\mu_{e_{n_1} \vee e_{n_2} \ldots \vee e_{n_k}}$ will have its support space as the smallest subspace of $H$ containing the vectors $e_{n_1}, e_{n_2}, \ldots, e_{n_k}$. So the support space of $\mu_{e_{n_1} \vee e_{n_2} \ldots \vee e_{n_k}}$ will be $V$ itself. This implies that $\mu \cong \mu_{e_{n_1} \vee e_{n_2} \ldots \vee e_{n_k}}$ and hence $\mu$ is generated by atomic VGFM$s$.

Note 3.20. It can be seen that corresponding to any finite dimensional VGFM, there is a unit vector $x \in H$ such that $\mu \cong \mu_x$.

Proof. Let $\alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_k}$, be real numbers such that

$$\alpha_{n_1}^2 + \alpha_{n_2}^2 + \cdots + \alpha_{n_k} = 1.$$ 

Let $x = \alpha_{n_1} \cdot e_{n_1} + \alpha_{n_2} \cdot e_{n_2} + \cdots + \alpha_{n_k} \cdot e_{n_k}$. Then $x$ is a unit vector in $H$ and the only orthonormal basis elements of $H$ that make non-zero inner product with $x$ are $e_{n_1}, e_{n_2}, \ldots, e_{n_k}$. So by the definitions of the support space and null space of a VGFM and the precedence, it follows that $\mu_{e_{n_1}} << \mu_x$, $\mu_{e_{n_2}} << \mu_x \ldots$ and $\mu_{e_{n_k}} << \mu_x$. As in the proof of 3.19, it follows that $\mu_{e_{n_1} \vee e_{n_2} \ldots \vee e_{n_k}} \cong \mu_x$.

References