Existence and Controllability Result for the Nonlinear First Order Fuzzy Neutral Integrodifferential Equations with Nonlocal Conditions

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\textit{ABSTRACT}

In this paper, we devoted study the existence and controllability for the nonlinear fuzzy neutral integrodifferential equations with control system in $E_N$. Moreover we study the fuzzy solution for the normal, convex, upper semicontinuous and compactly supported interval fuzzy number. The results are obtained by using the contraction principle theorem. An example to illustrate the theory.

\textit{KEYWORDS}

Controllability, Fixed Point Theorem, Integrodifferential Equations, Nonlocal Condition.

1. INTRODUCTION

First introduced fuzzy set theory Zadeh[15] in 1965. The term "fuzzy differential equation" was coined in 1978 by Kandel and Byatt[6]. This generalization was made by Puri and Ralescu [12] and studied by Kaleva [5]. It soon appeared that the solution of fuzzy differential equation interpreted by Hukuhara derivative has a drawback: it became fuzzier as time goes. Hence, the fuzzy solution behaves quite differently from the crisp solution. To alleviate the situation, Hullermeier[4] interpreted fuzzy differential equation as a family of differential inclusions. The main short coming of using differential inclusions is that we do not have a derivative of a fuzzy number valued function. There is another approach to solve fuzzy differential equations which is known as Zadeh’s extension principle (Misukoshi, Chalco-Cano, Román-Flores, Bassanezi[9]), the basic idea of the extension principle is: consider fuzzy differential equation as a deterministic differential equation then solve the deterministic differential equation. After getting deterministic solution, the fuzzy solution can be obtained by applying extension principle to deterministic solution. But in Zadeh’s extension principle we do not have a derivative of a fuzzy number valued function either. In [2], Bede, Rudas, and Bencsik[4], strongly generalized derivative concept was introduced. This concept allows us to solve the mentioned shortcomings and in Khastan et al. [7] authors studied higher order fuzzy differential equations with strongly generalized derivative concept. Recently, Gasilov et al. [3] proposed a new method to solve a fuzzy initial value problem for the fuzzy linear system of differential equations based on properties of linear transformations.
Fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in $E_N$ was presented by Y.C.Kwun et al. [8]. Recently, the above concept has been extended to the integrodifferential equations by J.H.Park et al. [12]. We will combine these earlier work and extended the study to the following nonlinear first order fuzzy neutral integrodifferential equations ($u(t) \equiv 0$) with nonlocal conditions:

$$\frac{d}{dt}(x(t) - h(t, x(t))) = A(t)[x(t) + \int_0^t G(t - s)x(s)ds] + f(t, x(t)) + u(t), \quad t \in J = [0, b]$$

(2)

$$x(0) + g(x) = x_0$$

(3)

where $A(t): J \to E_N$ is fuzzy coefficient, $E_N$ is the fuzzy set of all upper semicontinuous, convex, normal fuzzy numbers with bounded $\alpha-$ level intervals, $f: J \times E_N \to E_N$, $h: J \times E_N \to E_N$, $g: E_N \to E_N$ are all nonlinear functions, $G(t)$ is $n \times n$ continuous matrix such that $\frac{dG(t)x}{dt}$ is continuous for $x \in E_N$ and $t \in J$ with $\|G(t)\| \leq k$, $k > 0$, $u: J \to E_N$ is control function.

The rest of this paper is organized as follows. In section 2, some preliminaries are presented. In section 3, existence solution of fuzzy neutral integrodifferential equations. In section 4, we study on nonlocal control of solutions for the neutral system. In section 5, an example.

2. PRELIMINARIES

In section, we shall introduce some basic definitions, notations, lemmas and result which are used throughout this paper. A fuzzy subset of $\mathbb{R}^n$ is defined in terms of a membership function which assigns to each point $x \in \mathbb{R}^n$ a grade of membership in the fuzzy set. Such a membership function is denoted by

$$u: \mathbb{R}^n \to [0, 1].$$

Throughout this paper, we assume that $u$ maps $\mathbb{R}^n$ onto $[0, 1]$, $[u]^0$ is a bounded subset of $\mathbb{R}^n$, $u$ is upper semicontinuous, and $u$ is fuzzy convex. We denote by $E^n$ the space of all fuzzy subsets $u$ of $\mathbb{R}^n$ which are normal, fuzzy convex, and upper semicontinuous fuzzy sets with bounded supports. In particular, $E^1$ denotes the space of all fuzzy subsets $u$ of $\mathbb{R}$.

A fuzzy number $a$ in real line $\mathbb{R}$ is a fuzzy set characterized by a membership function $\chi_a$

$$\chi_a: \mathbb{R} \to [0, 1].$$

A fuzzy number $a$ is expressed as

$$a = \int_{\mathbb{R}} \chi_a$$
with the understanding that $\chi_a(x) \in [0,1]$, represents the grade of membership of $x$ in $a$ and $\bigcup$ denotes the union of $\chi_a$.

**Definition 2.1** A fuzzy number $a \in \mathbb{R}$ is said to be convex if, for any real numbers $x, y, z$ in $\mathbb{R}$ with $x \leq y \leq z$,

$$\chi_a(y) \geq \min\{\chi_a(x), \chi_a(z)\}$$

**Definition 2.2** If the height of a fuzzy set equals one, then the fuzzy set is called normal. Thus, a fuzzy number $a \in \mathbb{R}$ is called normal, if the followings holds:

$$\max_x \chi_a(x) = 1.$$

**Result 2.1** [10] Let $E_N$ be the set of all upper semicontinuous convex normal fuzzy numbers with bounded $\alpha$-level intervals. This means that if $a \in E_N$, then $\alpha$-level set

$$[a]^\alpha = \{x \in \mathbb{R} : a(x) \geq \alpha, 0 \leq \alpha \leq 1\},$$

is a closed bounded interval, which we denote by

$$[a]^\alpha = [a_q^\alpha, a_u^\alpha]$$

and there exists a $t_\alpha \in \mathbb{R}$ such that $a(t_\alpha) = 1$.

**Result 2.2** [10] Two fuzzy numbers $a$ and $b$ are called equal $a = b$, if $\chi_a(x) = \chi_b(x)$, for all $x \in \mathbb{R}$. It follows that

$$a = b \iff [a]^\alpha = [b]^\alpha, \text{ for all } \alpha \in (0,1].$$

**Result 2.3** [10] A fuzzy number $a$ may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha[a]^\alpha,$$

where $\alpha[a]^\alpha$ is the product of a scalar $\alpha$ with the set $[a]^\alpha$ and $\int$ is the union of $[a]^\alpha$ with $\alpha$ ranging from 0 to 1.

**Definition 2.3** A fuzzy number $a \in \mathbb{R}$ is said to be positive if $0 < a_1 < a_2$ holds for the support $\Gamma_a = [a_1, a_2]$ of $a$, that is, $\Gamma_a$ is in the positive real line. Similarly, $a$ is called negative if $a_1 < a_2 < 0$ and zero if $a_1 \leq 0 \leq a_2$. 

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Lemma: 2.1 [14] If $a, b \in E_N$, then for $\alpha \in (0, 1)$,

$[a + b]^{\alpha} = [a_q^{\alpha} + b_q^{\alpha}, a_r^{\alpha} + b_r^{\alpha}]$,

$[ab]^{\alpha} = [\min\{a_i^{\alpha}, b_i^{\alpha}\}, \max\{a_i^{\alpha}, b_i^{\alpha}\}]$, $(i, j = q, r)$,

$[a - b]^{\alpha} = [a_q^{\alpha} - b_q^{\alpha}, a_r^{\alpha} - b_r^{\alpha}]$.

Lemma: 2.2 [14] Let $[a_q^{\alpha}, a_r^{\alpha}]$, $0 < \alpha \leq 1$, be a given family of nonempty intervals. If

$[a_q^{\beta}, a_r^{\beta}] \subset [a_q^{\alpha}, a_r^{\alpha}]$ for $0 < \alpha \leq \beta$,

$[\lim\limits_{k \to \infty} a_q^{\alpha_k}, \lim\limits_{k \to \infty} a_r^{\alpha_k}] = [a_q^{\alpha}, a_r^{\alpha}]$,

whenever $(\alpha_k)$ is nondecreasing sequence converting to $\alpha \in (0, 1)$, then the family $[a_q^{\alpha}, a_r^{\alpha}]$, $0 < \alpha \leq 1$, are the $\alpha$-level sets of a fuzzy number $a \in E_N$.

Let $x$ be a point in $\mathbb{R}^n$ and $A$ be a nonempty subsets of $\mathbb{R}^n$. We define the Hausdroff separation of $B$ from $A$ by

$$d(x, A) = \inf \{\|x - a\| : a \in A\}.$$ 

Now let $A$ and $B$ be nonempty subsets of $\mathbb{R}^n$. We define the Hausdroff separation of $B$ from $A$ by

$$d_H^*(B, A) = \sup \{d(b, A) : b \in B\}.$$ 

In general,

$$d_H^*(A, B) \neq d_H^*(B, A).$$

We define the Hausdroff distance between nonempty subsets of $A$ and $B$ of $\mathbb{R}^n$ by

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\}.$$ 

This is now symmetric in $A$ and $B$. Consequently,

1. $d_H(A, B) \geq 0$ with $d_H(A, B) = 0$ if and only if $\overline{A} = \overline{B}$;

2. $d_H(A, B) = d_H(B, A)$;
3. \( d_H(A, B) \leq d_H(A, C) + d_H(C, B) \);

for any nonempty subsets of \( A, B \) and \( C \) of \( \mathbb{R}^n \). The Hausdroff distance is a metric, the Hausdroff metric.

The supremum metric \( d_\infty \) on \( E^n \) is defined by

\[
d_\infty(u, v) = \sup\{d_H([u]^{\alpha}, [v]^{\alpha}) : \alpha \in (0, 1]\}, \text{ for all } u, v \in E^n,
\]

and is obviously metric on \( E^n \).

The supremum metric \( H_1 \) on \( C(J, E^n) \) is defined by

\[
H_1(x, y) = \sup\{d_\infty(x(t), y(t)) : t \in J\}, \text{ for all } x, y \in C(J, E^n)\}.
\]

We assume the following conditions to prove the existence of solution of the equation (1.2).

(H1). The nonlinear function \( g : J \times E_N \rightarrow E_N \) is a continuous function and satisfies the inequality

\[
d_H([g(x(.)])^{\alpha}, [g(y(.)])^{\alpha}) \leq \delta_s d_H([x(.)]^{\alpha}, [y(.)]^{\alpha})
\]

(H2). The inhomogeneous term \( f : J \times E_N \rightarrow E_N \) is continuous function and satisfies a global Lipschitz

\[
d_H([f(s, x(s))]^{\alpha}, [f(s, y(s))]^{\alpha}) \leq \delta_f d_H([x(.)]^{\alpha}, [y(.)]^{\alpha})
\]

(H3). The nonlinear function \( h : J \times E_N \rightarrow E_N \) is continuous function and satisfies the global lipschitz condition

\[
d_H([h(s, x(.)])^{\alpha}, [h(s, y(.)])^{\alpha}) \leq \delta_h d_H([x(.)]^{\alpha}, [y(.)]^{\alpha})
\]

(H4). \( S(t) \) is the fuzzy number satisfies for \( y \in E_N \), \( S'y \in C(J, E_N) \cap C(J, E_N) \) the equation

\[
\frac{d}{dt}S(t)y = A(t)(S(t)y + \int_0^t G(t - s)S(s)yds)
\]

\[
= A(t)S(t)y + \int_0^t S(t - s)A(s)G(s)ds, \quad t \in J
\]

such that

\[
[S(t)]^{\alpha} = [S_q^{\alpha}, S_r^{\alpha}]
\]

and \( S_i^{\alpha}(t), i = q, r \) are continuous. That is, a postive constant \( \delta_s \) such that \( \|S_i^{\alpha}(t)\| < \delta_s \)

(H5) \((\Delta_1 + \Delta_2b)<1\), where \( \Delta_1 = \delta_s(\delta_f + \delta_h + \delta_s \delta_f) + \delta_h \) and \( \Delta_2 = \delta_f (\delta_f + M_h \delta_h) \)
3. EXISTENCE AND UNIQUENESS OF FUZZY SOLUTION

The solution of the equations (2)-(3) \((u \equiv 0)\) is of the form

\[
\gamma(t) = S(t)[x_0 - g(x) - h(0,x_0 - g(x))] + h(t,x(t)) + \int_0^t A(s)S(t-s)h(s,x(s))ds \\
+ \int_0^t S(t-s)f(s,x(s))ds
\]  

(4)

where \(\gamma\) is a continuous function from \(C(J;E_N)\) to itself.

**Theorem:** 3.1 Suppose that hypotheses (H1)-(H5) are satisfied. Then the equation (4) has unique fixed point in \(C(J;E_N)\).

**Proof.** For \(x, y \in C(J;E_N)\),

\[
d^\Delta([\gamma(t)]^a,[\gamma(t)]^a) \\
= d^\Delta([S(t)[x_0 - g(x) - h(0,x_0 - g(x))] + h(t,x(t)) \\
+ \int_0^t A(s)S(t-s)h(s,x(s))ds + \int_0^t S(t-s)f(s,x(s))ds]^a) \\
+ \int_0^t [S(t)[x_0 - g(y) - h(0,x_0 - g(y))] + h(t,y(t)) + \int_0^t A(s)S(t-s)h(s,y(s))ds \\
+ \int_0^t S(t-s)f(s,y(s))ds]^a)
\]

\[
\leq d^\Delta([S(t)[x_0 - g(x) - h(0,x_0 - g(x))]^a,[S(t)x_0 - g(y) - h(0,x_0 - g(y))]^a) \\
+ d^\Delta([h(t,x(t))]^a,[h(t,y(t))]^a) \\
+ d^\Delta([\int_0^t A(s)S(t-s)h(s,x(s))ds]^a,[\int_0^t A(s)S(t-s)h(s,y(s))ds]^a) \\
+ d^\Delta([\int_0^t S(t-s)f(s,x(s))ds]^a,[\int_0^t S(t-s)f(s,y(s))ds]^a) \\
\leq \delta_s d^\Delta([g(x) + h(0,x_0 - g(x))]^a,[g(y) + h(0,x_0 - g(y))]^a) + \delta_h d^\Delta([x]^a,[y]^a) \\
+ \delta_s M \int_0^t d^\Delta([h(s,x(s))]^a,[h(s,y(s))]^a)dt + \delta_s \int_0^t d^\Delta([f(s,x(s))]^a,[f(s,y(s))]^a)dt \\
\leq \delta_s (\delta_h + \delta_h^s) + \delta_h d^\Delta([x]^a,[y]^a) \\
+ \delta_s (\delta_f + M \delta_h) \int_0^t d^\Delta([x]^a,[y]^a)dt \\
= \Delta_1 d^\Delta([x(t)]^a,[y(t)]^a) + \Delta_2 \int_0^t d^\Delta([x(t)]^a,[y(t)]^a)dt
\]

where, \(\Delta_1 = \delta_s (\delta_h + \delta_h^s) + \delta_h\) and \(\Delta_2 = \delta_s (\delta_f + M \delta_h)\).

Therefore,

\[
d^a(\gamma(t), \eta(t)) = \sup_{a \in [0,1]} d^\Delta([\gamma(t)]^a,[\eta(t)]^a)
\]
\[ \leq \Delta_1 \sup_{\alpha \in [0,1]} d_{\alpha} \left( \left\{ x(t) \right\}^\alpha, \left\{ y(t) \right\}^\alpha \right) \]
\[ + \Delta_2 \sup_{\alpha \in [0,1]} \int_0^t d_{\alpha} \left( \left\{ x(t) \right\}^\alpha, \left\{ y(t) \right\}^\alpha \right) dt \]
\[ \leq \Delta_1 d_{\alpha}(x(t), y(t)) + \Delta_2 \int_0^t d_{\alpha}(x(t), y(t)) dt \]

Hence,
\[ H_1(\mathcal{X}(t), \mathcal{Y}(t)) = \sup_{\tau \in J} d_{\alpha}(\mathcal{X}(t), \mathcal{Y}(t)) \]
\[ \leq \Delta_1 \sup_{\tau \in J} d_{\alpha}(x(t), y(t)) + \Delta_2 \int_0^t \sup_{\tau \in [0,b]} d_{\alpha}(x(\tau), y(\tau)) d\tau \]
\[ \leq \Delta_1 H_1(x(t), y(t)) + \Delta_2 b H_1(x(t), y(t)) \]
\[ = (\Delta_1 + \Delta_2 b) H_1(x(t), y(t)) \]

Then by hypotheses, \( \gamma \) is a contraction mapping. By using Banach fixed point theorem, equations (2)-(3) have a unique fixed point, \( x \in C(J, E_N) \)

### 4. CONTROLLABILITY OF FUZZY SOLUTION

In this section, we show for the controller term in equation(2)-(3), and the solution of the form
\[ x(t) = S(t)[x_0 - g(x) - h(0, x_0 - g(x))] + h(t, x(t)) + \int_0^t S(t-s)A(s)h(s, x(s))ds \]
\[ + \int_0^t S(t-s)f(s, x(s))ds + \int_0^t S(t-s)u(s)ds \]  \hspace{1cm} (5)

**Definition 4.1** [13] The equation (4.1) is controllable if there exists \( u(t) \) such that the fuzzy solution \( x(t) \) of (5) satisfies \( x(b) = x^1 - g(x) \), that is \( \left\{ x(b) \right\}^\alpha = \left\{ x^1 - g(x) \right\}^\alpha \), where \( x^1 \) is a target set.

The linear controll system is nonlocal controllable. That is,
\[ x(b) = S(b)[x_0 - g(x)] + \int_0^b S(b-s)u(s)ds \]
\[ = x^1 - g(x) \]
\[ \left\{ x(b) \right\}^\alpha = \left[ S(b)[x_0 - g(x)] + \int_0^b S(b-s)u(s)ds \right]^\alpha \]
\[ = [S^\alpha_q(b)[x_0 - g^\alpha_q(x)] + \int_0^b S^\alpha_q(b-s)u^\alpha_q(s)ds] \]
\[ = [(x^1) - g^\alpha_q(x), (x^1)]^\alpha = [x^1 - g(x)]^\alpha. \]

We Define new fuzzy mapping \( \zeta : P(R) \rightarrow E_N \) by
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\[ \xi^\alpha(v) = \begin{cases} \int_0^v S(t-s)v(s) \, ds, & v \in \Gamma_u, \\ 0, & \text{otherwise} \end{cases} \]

Then there exists \( \zeta_i^\alpha (i = q, r) \) such that

\[ \zeta_q^\alpha (v_q) = \int_0^v S_q^\alpha(t-s)v_q(s) \, ds, \quad v_q \in [u_q^\alpha, u^1_q] \]

\[ \zeta_r^\alpha (v_r) = \int_0^v S_r^\alpha(t-s)v_r(s) \, ds, \quad v_r \in [u^1, u_r^\alpha] \]

We assume that \( \zeta_i^\alpha \)'s are bijective mappings. Hence the \( \alpha \)-set of \( u(s) \) are

\[ [u(s)]^\alpha = [u_q^\alpha(s), u_r^\alpha(s)] \]

\[ = [(\zeta_q)^{-1}((x_1)^\alpha_q - g_q^\alpha(x) - S_q^\alpha(b)(x_0)_q^\alpha - g_q^\alpha(0, x_0 - g(x))] \]

\[ - h_q^\alpha(t, x(t)) - \int_0^t S_q^\alpha(t-s)A_q^\alpha(s)h_q^\alpha(t, x(s)) \, ds \]

\[ - \int_0^t S_q^\alpha(t-s)f_q^\alpha(t, x(s)) \, ds, \]

\[ (\zeta_r)^{-1}((x_1)^\alpha_r - g_r^\alpha(x) - S_r^\alpha(b)(x_0)_r^\alpha - g_r^\alpha(0, x_0 - g(x))] \]

\[ - h_r^\alpha(t, x(t)) - \int_0^t S_r^\alpha(t-s)A_r^\alpha(s)h_r^\alpha(t, x(s)) \, ds \]

\[ - \int_0^t S_r^\alpha(t-s)f_r^\alpha(t, x(s)) \, ds \]

Then substituting this expression into equation (5) yields \( \alpha \)-level set of \( x(b) \)

\[ [x(b)]^\alpha \]

\[ = [S_q^\alpha(b)(x_0)_q^\alpha - g_q^\alpha(x) - S_q^\alpha(b)(x_0)_q^\alpha - h_q^\alpha(s, x(s)) \]

\[ + \int_0^b S_q^\alpha(b-s)A_q^\alpha(s)h_q^\alpha(t, x(s)) \, ds + \int_0^b S_q^\alpha(b-s)f_q^\alpha(s, x(s)) \, ds \]

\[ + \int_0^b S_q^\alpha(b-s)((x_1)^\alpha_q - g_q^\alpha(x) - S_q^\alpha(b)(x_0)_q^\alpha) \]

\[ - g_q^\alpha(x) - h_q^\alpha(0, x_0)] - h_q^\alpha(s, x(s)) \]

\[ - \int_0^b S_q^\alpha(b-s)A_q^\alpha(s)h_q^\alpha(s, x(s)) \, ds - \int_0^b S_q^\alpha(b-s)f_q^\alpha(s, x(s)) \, ds, \]

\[ S_r^\alpha(b)(x_0)_r^\alpha - g_r^\alpha(x) - h_r^\alpha(0, x_0 - g(x))] \]

\[ + h_r^\alpha(t, x(t)) + \int_0^b S_r^\alpha(b-s)A_r^\alpha(s)h_r^\alpha(s, x(s)) \, ds + \int_0^b S_r^\alpha(b-s)f_r^\alpha(s, x(s)) \, ds \]

\[ + \int_0^b S_r^\alpha(b-s)((x_1)^\alpha_r - g_r^\alpha(x) - S_r^\alpha(b)(x_0)_r^\alpha) \]

\[ - g_r^\alpha(x) - h_r^\alpha(0, x_0 - g(x))] \]
\[-h^\alpha_t(s, x(s)) - \int_0^b S^\alpha_r(b - s) A^\alpha_q(s) h^\alpha_f(s, x(s))\,ds\]
\[-\int_0^b S^\alpha_r(b - s) f^\alpha_r(s, x(s))\,ds\]

\[= \left[ S^\alpha_q(b)[(x_0)^\alpha_q - g^\alpha_q(x) - h^\alpha_q(0, x_0 - g(x))] + h(s, x(s)) + \int_0^b S^\alpha_q(b - s) A^\alpha_q(s) h^\alpha_q(s, x(s))\,ds\right.\]
\[+ \int_0^b S^\alpha_q(b - s) f^\alpha_q(s, x(s))\,ds + \zeta^\alpha_q(\xi^\alpha_q)^{-1}((x^1)^\alpha_q - g^\alpha_q(x) - S^\alpha_q(b)[(x_0)^\alpha_q - g^\alpha_q(x) - h^\alpha_q(0, x_0 - g(x))] - h^\alpha_q(s, u(s)) - \int_0^b S^\alpha_q(b - s) A^\alpha_q(s) h^\alpha_q(s, x(s))\,ds\]
\[\left. - \int_0^b S^\alpha_q(b - s) f^\alpha_q(s, x(s))\,ds\right] \]
\[= [(x^1)^\alpha_q - g^\alpha_q(x), (x^1)^\alpha_r - g^\alpha_r(x)]\]
\[= [x^1 - g(x)]^\alpha.\]

We now set
\[\Omega x(t) = S(t)[x_0 - g(x) - h(0, x_0 - g(x))] + h(t, u(t)) + \int_0^t S(t - s) A(s) h(t, x(s))\,ds\]
\[+ \int_0^t S(t - s) f(s, x(s))\,ds\]
\[+ \int_0^t S(t - s) \xi^-1(x^1 - g(x) - S(b)[x_0 - g(x) - h(0, x_0 - g(x))])\]
\[- h(t, x(t)) - \int_0^t S(b - s) A(s) h(s, x(s))\,ds - \int_0^t S(b - s) f(s, x(s))\,ds\]

where the fuzzy mapping \(\xi^{-1}\) satisfied above statement.

Now notice that \(\Omega x(T) = x^1 - g(x)\), which means that the control \(u(t)\) steers the equation (5) from the origin to \(x^1\) in the time \(b\) provided we can obtain a fixed point of the nonlinear operator \(\Omega\).

Assume that the hypotheses

(H6) The system (4.1) is linear \(f \equiv 0\) is nonlocal controllable.

(H7) \((\delta_n + (\delta_f + \delta_b + \delta_n) \leq 1.\)
**Theorem:** 4.1 Suppose that the hypotheses (H1)-(H7) are satisfied. Then the equation (5) is a nonlocal controllable.

**Proof.** We can easily check that $\Omega$ is continuous from $C([0,b]:E_N)$ to itself. For $x, y \in C([0,b]:E_N)$,

$$d_H([\Omega x(t)]^\alpha,[\Omega y(t)]^\alpha)$$

$$= d_H (\{S(t)[x_0 - g(x) - h(0,x_0 - g(x))] + h(t, x(t)) + \int_0^b S(b-s) A(s) h(s, y(s)) ds + \int_0^b S(t-s) f(s, x(s)) ds$$

$$+ \int_0^b S(b-s) \frac{1}{\xi} (x^l - g(x) - S(b)[x_0 - g(x)] - h(0,x_0 - y(y)) - h(t, x(t)) - \int_0^b S(b-s) A(s) h(s, y(s)) ds$$

$$- \int_0^b S(b-s) f(s, x(s)) ds,$$

$$[S(t)[x_0 - g(y) - h(0,x_0 - g(y))] + h(t, y(t)) + \int_0^b S(t-s) A(s) h(s, y(s)) ds$$

$$+ \int_0^b S(t-s) f(s, y(s)) ds$$

$$+ \int_0^b S(b-s) \frac{1}{\xi} (x^l - S(b-s)[x_0 - g(y) - h(0,x_0 - y(y)) - h(t, y(t)) - \int_0^b S(b-s) A(s) h(s, y(s)) ds$$

$$- \int_0^b S(b-s) f(s, y(s)) ds)]^\alpha)$$

$$\leq d_H ([S(t)[g(x) + h(0,x_0 - g(x))] + h(t, x(t)) + \int_0^b S(t-s) A(s) h(s, x(s)) ds$$

$$+ \int_0^b S(t-s) f(s, x(s)) ds + \sum_{0<\tau \in I_k} S(t_k - t) I_k (x(t_k^-))]^\alpha,$$

$$[S(t)[g(y) + h(0,x_0 - g(y))] + h(t, y(t)) + \int_0^b S(t-s) A(s) h(s, y(s)) ds$$

$$+ \int_0^b S(t-s) f(s, y(s)) ds$$

$$+ d_H (\{S(t-s) \frac{1}{\xi} (x^l - g(x) - S(b)[g(x) + h(0,x_0 - g(x))] - h(s, x(s))$$

$$- \int_0^b S(t-s) A(s) h(s, x(s)) ds - \int_0^b S(b-s) f(s, x(s)) ds,$$

$$[S(t-s) \frac{1}{\xi} (x^l - g(y) - S(b)[g(y) + h(0,x_0 - g(y))] - h(s, y(s))$$

$$- \int_0^b S(b-s) A(s) h(s, y(s)) ds - \int_0^b S(b-s) f(s, y(s)) ds$$

$$\leq (\delta_1 (\delta_2 + \delta_3 + \delta_4 \delta_5) + \delta_6) d_H ([x(s)]^\alpha,[y]^\alpha)$$
Let $\kappa_1 = \delta_x (\delta_g + \delta_h)$, and $\kappa_2 = \delta_x (M \delta_h)$, then we have

$$
\leq 2 \kappa_1 d_H ([x(s)]^\alpha, [y]^\alpha) + \kappa_2 \left( \int_0^b d_H ([x(s)]^\alpha, [y(s)]^\alpha) + \int_0^b d_H ([x(s)]^\alpha, [y(s)]^\alpha) \right)
$$

Therefore

$$
d_{\infty} (\Omega x(t), \Omega y(t)) = \sup_{\alpha \in [0,1]} d_H ([\Omega x(t)]^\alpha, [\Omega y(t)]^\alpha)
$$

$$
\leq 2 \kappa_1 d_{\infty} ([x(s)]^\alpha, [y]^\alpha)
$$

$$
+ \kappa_2 \left( \int_0^b d_{\infty} ([x(s)]^\alpha, [y(s)]^\alpha) + \int_0^b d_{\infty} ([x(s)]^\alpha, [y(s)]^\alpha) \right)
$$

Hence

$$
H_1 (\Omega x(t), \Omega y(t)) = \sup_{\alpha \in [0,1]} d_H ([\Omega x(t)]^\alpha, [\Omega y(t)]^\alpha)
$$

$$
\leq (2 \kappa_1 + 2 \kappa_2 b) H_1 (x, y)
$$

$$
= (2(\kappa_1 + \kappa_2 b) H_1 (x, y)
$$

By hypotheses $(H_b)$, we take sufficiently small $b$, $\Omega$ is a contraction mapping. By Banach fixed point theorem, equation (5) has a unique fixed point $x \in C([0, b]: E_N)$.

5.EXAMPLE

Consider the fuzzy solution of the nonlinear fuzzy neutral integrodifferential equation of the form:

$$
\frac{d}{dt} \left( x(t) - 2tx(t)^2 \right) = 2 \left( x(t) - \int_0^t e^{-t-s} x(s) ds \right) + 3tx(t)^2 + u(t), \quad t \in J, \quad (6)
$$

$$
x(0) + \sum_{k=1}^n c_k x(t_k) = 0 \in E_N, \quad (7)
$$

where $x^*$ is target set, and the $\alpha$ - level set of fuzzy number $0, 2$ and $3$ are

$[0]^\alpha = [\alpha - 1, 1 + \alpha]$, for $\alpha \in [0, 1]$

$[2]^\alpha = [\alpha + 1, 3 - \alpha]$, for $\alpha \in [0, 1]$

$[3]^\alpha = [\alpha + 2, 4 - \alpha]$, for $\alpha \in [0, 1]$

Let $G(t-s) = e^{-t-s}$, $f(t, u(t)) = 3tu(t)^2$, $h(t, u(t)) = 2tu(t)^2$. Then $\alpha$ - level set of $g(x) = \sum_{k=1}^n c_k x(t_k)$ is
\[ [g(x)]^\alpha = \left[ \sum_{k=1}^{n} c_k x(t_k) \right]^\alpha \]
\[ = \left[ \sum_{k=1}^{n} c_k^\alpha x(t_k), \sum_{k=1}^{n} c_k^\alpha x(t_k) \right] \]
\[ d_H([g(x)]^\alpha),[g(y)]^\alpha) = d_H([\sum_{k=1}^{n} c_k x(t_k)]^\alpha,[\sum_{k=1}^{n} c_k y(t_k)]^\alpha) \]
\[ = d_H([\sum_{k=1}^{n} c_k x(t_k),\sum_{k=1}^{n} c_k x(t_k)]],[\sum_{k=1}^{n} c_k y(t_k),\sum_{k=1}^{n} c_k y(t_k)]) \]
\[ \leq \| c_k \| \max \{ |x_q^\alpha(t_k) - y_q^\alpha(t_k)|, |x_r^\alpha(t_k) - y_r^\alpha(t_k)| \} \]
\[ \leq \delta d_H([x(t)]^\alpha,[y(t)]^\alpha) \]

where, \[ \delta = \| c_k \| \]

and \[ [3]^\alpha = [\alpha + 2,4 - \alpha], \text{ for } \alpha \in [0,1] \text{ and the } \alpha \text{ - level set of } f(t,x(t)) \text{ is} \]
\[ [f(t,x(t))]^\alpha = [3tx(t)^2]^\alpha \]
\[ = t[3]^\alpha[x(t)^2]^\alpha \]
\[ = t[(\alpha + 2)(x_q^\alpha(t)^2),(4 - \alpha)(x_r^\alpha(t)^2)] \]
\[ d_H([f(t,x(t))]^\alpha),[f(t,y(t))]^\alpha) \]
\[ = d_H(t[(\alpha + 2)(x_q^\alpha(t)^2),(4 - \alpha)(x_r^\alpha(t)^2)],t[(\alpha + 2)(y_q^\alpha(t)^2),(4 - \alpha)(y_r^\alpha(t)^2)]) \]
\[ = t \max \{ (\alpha + 2)[(x_q^\alpha(t)^2) - (y_q^\alpha(t)^2)],(4 - \alpha)[(x_r^\alpha(t)^2) - (y_r^\alpha(t)^2)] \} \]
\[ \leq b(4 - \alpha) \max \{ |(x_q^\alpha(t)) - (y_q^\alpha(t))|,|(x_q^\alpha(t)) + (y_q^\alpha(t))|,|(x_r^\alpha(t)) - (y_r^\alpha(t))| |(x_r^\alpha(t)) + (y_r^\alpha(t))| \} \]
\[ \leq 4b \max \{ |x_q^\alpha(t)| + |y_q^\alpha(t)| \max \{ |(x_q^\alpha(t)) - (y_q^\alpha(t))|,|(x_r^\alpha(t)) - (y_r^\alpha(t))| \} \]
\[ \leq \delta d_H([x(t)]^\alpha,[y(t)]^\alpha) \]

where, \[ \delta = 4b \max \{ |x_q^\alpha(t)| + |y_q^\alpha(t)| \]

\[ [3]^\alpha = [\alpha + 2,4 - \alpha], \text{ for } \alpha \in [0,1] \text{ and the } \alpha \text{ - level set of } h(t,x(t)) \text{ is} \]
\[ [h(t,x(t))]^\alpha = [2tx(t)^2]^\alpha \]
\[ = t[2]^\alpha[x(t)^2]^\alpha \]
\[ = t[(\alpha + 1,3 - \alpha)[(x_q^\alpha(t)^2),(x_r^\alpha(t)^2))] \]
\[ = t[(\alpha + 1)(x_q^\alpha(t)^2),(3 - \alpha)(x_r^\alpha(t)^2)] \]

we introduced the \[ \alpha \text{ - set of Equation (5.1) - (5.2) Thus,} \]
\[ d_H([h(t,x(t))]^\alpha),[h(t,y(t))]^\alpha) \]
\[ \begin{align*}
&= d_H(t[(\alpha + 1)(x_q^\alpha(t))^2, (3-\alpha)(x_r^\alpha(t))^2], t[(\alpha + 1)(y_q^\alpha(t))^2, (3-\alpha)(y_r^\alpha(t))^2]) \\
&= \max\{t(\alpha+1) | (x_q^\alpha(t))^2 - (y_q^\alpha(t))^2 |, t(3-\alpha) | (x_r^\alpha(t))^2 - (y_r^\alpha(t))^2 | \}
\end{align*} \]

\[ \leq b(3-\alpha) \max\{ |(x_q^\alpha(t)) - (y_q^\alpha(t))|, |(x_q^\alpha(t)) + (y_q^\alpha(t))|, |(x_r^\alpha(t)) - (y_r^\alpha(t))|, |(x_r^\alpha(t)) + (y_r^\alpha(t))| \} \]

\[ \leq 3b \max\{ |(x_q^\alpha(t)) - (y_q^\alpha(t))|, |(x_q^\alpha(t)) + (y_q^\alpha(t))|, |(x_r^\alpha(t)) - (y_r^\alpha(t))|, |(x_r^\alpha(t)) + (y_r^\alpha(t))| \} \]

\[ = \delta_g d_H([x(t)]^\alpha, [y(t)]^\alpha) \]

where, \( \delta_g = 3b \max\{ |(x_q^\alpha(t)) - (y_q^\alpha(t))|, |(x_q^\alpha(t)) + (y_q^\alpha(t))|, |(x_r^\alpha(t)) - (y_r^\alpha(t))|, |(x_r^\alpha(t)) + (y_r^\alpha(t))| \} \]

Next we prove the nonlocal controllability parts, let us take target set, \( x^1 = 2 \)

\[ [u(s)]^\alpha = [u_q^\alpha, u_r^\alpha] \]

\[ = \left[ x_q^\alpha(t_k) - t(\alpha + 1)(x_q^\alpha(t))^2 \right] \]

\[ - \int_0^b S_q^\alpha(b-s) (\alpha + 1) (x_q^\alpha(s))^2 ds - \int_0^b S_q^\alpha(b-s) (\alpha + 2) (x_q^\alpha(s))^2 ds \]

\[ - \int_0^b S_q^\alpha(b-s) (\alpha + 1) (x_r^\alpha(s))^2 ds, \xi(t) = ((3-\alpha) - \sum_{k=1}^n c_k x_r^\alpha(t_k)) \]

\[ - t(\alpha + 1)(x_r^\alpha)^2 - \int_0^b S_q^\alpha(b-s) (\alpha + 1) (x_r^\alpha)^2 ds - \int_0^b S_q^\alpha(b-s) (\alpha + 2) (x_r^\alpha)^2 ds \]

Then substituting this expression into the integral system with respect to (6)-(7) yields \( \alpha \) - level set of \( x(b) \).

\[ [x(b)]^\alpha = [S_q^\alpha(b)((\alpha - 1) - \sum_{k=1}^n c_k x(t_k))] + t(\alpha + 1)(x_q^\alpha(t))^2(t) + \int_0^b S_q^\alpha(t-s) t(\alpha + 1)(x_q^\alpha)^2 ds \]

\[ + \int_0^b S_q^\alpha(t-s) t(\alpha + 2)(x_q^\alpha)^2 ds + \int_0^b S_q^\alpha(t-s) t(\alpha + 1)(x_q^\alpha)(t)^2 ds \]

\[ + \int_0^b S_q^\alpha(b-s) (\alpha + 1) (x_q^\alpha)^2(t) - \int_0^b S_q^\alpha(b-s) t(\alpha + 1)(x_q^\alpha)^2(t) ds \]

\[ + \int_0^b S_q^\alpha(t-s) t(\alpha + 2)(x_q^\alpha)^2(s) ds - \int_0^b S_q^\alpha(t-s) t(\alpha + 1)(x_q^\alpha(s))^2 ds ds \]

\[ [S_q^\alpha(b)(1-\alpha)] t(\alpha + 1)(x_q^\alpha)^2(t) + \int_0^b S_q^\alpha(b-s)(3-\alpha)(x_q^\alpha)^2(t) ds \]
\[
+ \int_0^1 S^\alpha_r (t-s)t(4-\alpha)x^\alpha_r(s)^2 ds + \int_0^1 S^\alpha_s (t-s)t(3-\alpha)x^\alpha_s(s)^2(t) ds \\
+ \int_0^b S^\alpha_r (b-s)((3-\alpha)-t(\alpha+1)x^\alpha_r(s)^2(t) - \int_0^b S^\alpha_s (b-s)t(3-\alpha)x^\alpha_s(s)^2(t) ds \\
- \int_0^b S^\alpha_r (b-s)t(4-\alpha)x^\alpha_r(s)^2(s) ds - \int_0^b S^\alpha_s (b-s)t(3-\alpha)x^\alpha_s(s)^2 ds) ds) \\
= [(\alpha+1) - \sum_{k=1}^n c_k x^\alpha_r(t_k), (3-\alpha) - \sum_{k=1}^n c_k x^\alpha_s(t_k)] \\
= [2 - \sum_{k=1}^n c_k x(t_k)]^\alpha \\
= [x^1 - g(x)]
\]

Then all condition stated in theorem 4.1 are satisfied, so the system (6)-(7) is nonlocal controllable on [0, b].

6. CONCLUSION

In this paper, by using the concept of fuzzy number in \( E_N \), we study the existence and controllability for the nonlinear fuzzy neutral integrodifferential with nonlocal controll system in \( E_N \) and find the sufficient conditions of controllability for the controll system (2)-(3).

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REFERENCES


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