Properties of Fuzzy Inner Product Spaces

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Abstract

In this paper, natural inner product structure for the space of fuzzy n-tuples is introduced. Also we have introduced the ortho vector, stochastic fuzzy vectors, ortho-stochastic fuzzy vectors, ortho-stochastic fuzzy matrices and the concept of orthogonal complement of fuzzy vector subspace of a fuzzy vector space.

Keywords

Ortho vector, Stochastic vector, Ortho-Stochastic vector, Orthogonal Complement, Ortho-stochastic matrix, Reflection.

1. Introduction

A fuzzy vector space \( V_n \) is the set of all \( n \)-tuples \((x_1, x_2, \ldots, x_n)\) of a fuzzy algebra \( \mathfrak{A} = [0,1] \). The elements of \( V_n \) possess a natural linear space-like structure, where the elements are called fuzzy vectors. Also, we define an operation on \( V_n \) which is analogous to an inner product and then we can define a norm and orthogonality relation on \( V_n \). The concept of fuzzy inner product has been introduced by several authors like Ahmed and Hamouly [1], Kohli and Kumar [5], Biswas [4], etc. Also the notion of fuzzy norm on a linear space was introduced by Katsaras [9]. Later on many other mathematicians like Felbin [3], Cheng and Mordeso [10], Bag and Samanta [6], etc, have given different definitions of fuzzy normed spaces. In recent past lots of work have been done in the topic of fuzzy functional analysis, but only a few works have been done on fuzzy inner product spaces. Biswas [4] tried to give a meaningful definition of fuzzy inner product space and associated fuzzy norm functions. Later on, a modification of the definition given by Biswas [4] was done by Kohli and Kumar [5] and they also introduced the notion of fuzzy co-inner product space. But still there is no useful definition of fuzzy inner product available to work with in the sense that not much development has yet been found in fuzzy inner product (or fuzzy Hilbert) space with these definitions.

The concept of ortho-vector, stochastic vector, stochastic subspace, isometry, reflection in Boolean inner product space was introduced by Gudder and Latremoliere [2]. In this paper, we modify these concepts on a fuzzy inner product spaces. We have also developed the concept of isometry, isomorphism for fuzzy inner product space. Also, some basic properties and results
such as orthogonality, the orthogonal complement and the ortho-stochastic subspace are given and established.

2. PRELIMINARIES

In this article, for any \( a \in \mathbb{S} \), we denote \( a^c \) its complement. For any two elements \( a, b \in \mathbb{S} \), we denote the infimum of \( a \) and \( b \) by \( a \wedge b \) or by \( ab \) and the supremum of \( a \) and \( b \) by \( a \vee b \) or by \( a + b \). If \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) are in \( V_n \) and \( c \in \mathbb{S} \); then \( a + b = (a_1 \lor b_1, a_2 \lor b_2, \ldots, a_n \lor b_n) \) and \( ca = (c \land a_1, c \land a_2, \ldots, c \land a_n) \).

Throughout this paper we use the following definitions.

Definition 1 (Ortho-vector) A fuzzy vector \( x = (x_1, x_2, \ldots, x_n) \) is said to be an ortho-vector if \( x_i x_j = 0 \) for \( i, j = 1, 2, 3, \ldots, n \) and \( i \neq j \).

This implies that at most one coordinate of \( x \) is non-zero.

Example 1 The vector \( x = (0.5,0,0,\ldots,0) \) is an ortho-vector. The null vector \( 0 = (0,0,0,\ldots,0) \) is an ortho-vector.

Definition 2 (Stochastic-vector) A fuzzy vector \( x = (x_1, x_2, \ldots, x_n) \) is said to be a stochastic-vector if \( \sum_{i=1}^{n} x_i = 1 \).

Example 2 \( x = (0.5,0.1,0.9,0.5,0.2,1) \) is a stochastic vector in \( V_6 \) because \( \max\{0.5,0.1,0.9,0.5,0.2,1\} = 1 \).

Definition 3 (Inner Product) Let \( x = (x_1, x_2, \ldots, x_n) \), and \( y = (y_1, y_2, \ldots, y_n) \) in \( V_n \). Then the inner product of \( x \) and \( y \) is denoted by \( \langle x, y \rangle \) and is defined by \( \langle x, y \rangle = \sum_{i=1}^{n} (x_i \land y_i) \).

Definition 4 (Norm) Let \( x = (x_1, x_2, \ldots, x_n) \) in \( V_n \). Then the norm of \( x \) is denoted by \( \|x\| \) and is defined by \( \|x\| = \langle x, x \rangle \). The vector \( x \) is said to be a unit vector if \( \|x\| = 1 \). If we replace the scalar sums and products by the supremum and infimum in \( \mathbb{S} \), then the usual properties of the Euclidean inner product are satisfied by fuzzy inner product. Let \( x, y, z \in V_n \) and \( \alpha \in \mathbb{S} \); then

(i) \( \langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle \),

(ii) \( \langle x, y \rangle = \langle y, x \rangle \),

(iii) \( \langle \alpha x, y \rangle = \langle x, \alpha y \rangle = \alpha \langle x, y \rangle \),

(iv) \( \langle x, x \rangle = 0 \) if and only if \( x = (0,0,0,\ldots,0) = 0 \).
Definition 5 (Orthogonal Vector) Two fuzzy vectors $\underline{x}$ and $\underline{y}$ in $V_n$ are said to be orthogonal if 
$\langle \underline{x}, \underline{y} \rangle = 0$. In this case, we shall write $\underline{x} \perp \underline{y}$.

Example 3 Let $\underline{x} = (0,0.1,0.2,0.5)$ and $\underline{y} = (0.4,0.0,3.0,0.0)$ in $V_5$. Here $\langle \underline{x}, \underline{y} \rangle = 0$. Thus, $\underline{x} \perp \underline{y}$.

Now, we introduced a definition of orthogonal and orthonormal set of fuzzy vectors.

Definition 6 (Orthogonal and Orthonormal Set) Let $E \subseteq V_n$. Then $E$ is said to be an orthogonal set if $\langle \underline{x}, \underline{y} \rangle = 0$ for all $\underline{x}, \underline{y} \in E$ with $\underline{x} \neq \underline{y}$.

Also, $E \subseteq V_n$ is said to be an orthonormal set if $E$ is orthogonal i.e, $\langle \underline{x}, \underline{y} \rangle = 0$ for all $\underline{x}, \underline{y} \in E$ with $\underline{x} \neq \underline{y}$ and $\|\underline{x}\| = 1$ for all $\underline{x} \in E$.

Theorem 1 Let $\underline{x}, \underline{y} \in V_n$ and $c \in \mathbb{S}$. Then
(i) $\|c\underline{x}\| = c\|\underline{x}\|$.
(ii) $\|\underline{x} + \underline{y}\| = \|\underline{x}\| \vee \|\underline{y}\|$.
(iii) $\langle \underline{x}, \underline{y} \rangle \leq \|\underline{x}\| \wedge \|\underline{y}\|$.

Proof. (i) $\|c\underline{x}\| = \langle c\underline{x}, c\underline{x} \rangle = c\langle \underline{x}, \underline{x} \rangle = (c \wedge c)\langle \underline{x}, \underline{x} \rangle = c\langle \underline{x}, \underline{x} \rangle = c\|\underline{x}\|$ for all $\underline{x} \in V_n$.

(ii) $\|\underline{x} + \underline{y}\| = \bigvee_{i=1}^{n} (\underline{x}_i \vee \underline{y}_i) = (\bigvee_{i=1}^{n} \underline{x}_i) \vee (\bigvee_{i=1}^{n} \underline{y}_i) = \|\underline{x}\| \vee \|\underline{y}\|$.

(iii) $\langle \underline{x}, \underline{y} \rangle = \bigvee_{i=1}^{n} (\underline{x}_i \wedge \underline{y}_i) \leq \bigvee_{i,j=1}^{n} (\underline{x}_i \wedge \underline{y}_j) = (\bigvee_{i=1}^{n} \underline{x}_i) \wedge (\bigvee_{j=1}^{n} \underline{y}_j) = \|\underline{x}\| \wedge \|\underline{y}\|$.

3. PROPERTIES OF FUZZY ORTHO-STOCHASTIC VECTORS

Based on these definitions some results are established. Now we discuss the conditions for which two fuzzy vectors to be ortho-stochastic.

Theorem 2 If $\underline{x}$ and $\underline{y}$ are two stochastic vector in $V_n$, then $\langle \underline{x}, \underline{y} \rangle = 1$ if $\underline{x} = \underline{y}$, but the converse need not be true.

Proof. Let $\underline{x} = \underline{y}$.

Thus, $x_i = y_i$ for $i \in \{1,2,\ldots, n\}$.

Since $\underline{x}$ and $\underline{y}$ are stochastic, so, $x_i = 1$ and $y_i = 1$ for some $i \in \{1,2,\ldots, n\}$. Thus, 
$\langle \underline{x}, \underline{y} \rangle = \bigvee_{i=1}^{n} (x_i \wedge y_i) = 1$.

But, if we take $\underline{x} = (0.5,0.2,1)$ and $\underline{y} = (0.3,0.5,1)$, then $\underline{x}$ and $\underline{y}$ both are stochastic vector with $\langle \underline{x}, \underline{y} \rangle = 1$ and $\underline{x} \neq \underline{y}$. Thus, $\langle \underline{x}, \underline{y} \rangle = 1$ need not implies $\underline{x} = \underline{y}$. 

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Corollary 1 If \( \underline{x} \) and \( \underline{y} \) are two ortho-stochastic vector in \( V_n \), then \( \langle \underline{x}, \underline{y} \rangle = 1 \) if and only if \( \underline{x} = \underline{y} \).

**Proof.** Let \( \underline{x}, \underline{y} \) be two ortho-stochastic vectors in \( V_n \) such that \( \langle \underline{x}, \underline{y} \rangle = 1 \).

Taking \( \underline{x} = (x_1, x_2, \ldots, x_n) \) and \( \underline{y} = (y_1, y_2, \ldots, y_n) \) in \( V_n \). Since \( \underline{x}, \underline{y} \) are two ortho-vectors, thus, at most one coordinate of \( \underline{x}, \underline{y} \) is non-zero.

Now, \( \langle \underline{x}, \underline{y} \rangle = 1 \)
\[ \Rightarrow \bigvee_{i=1}^n (x_i \wedge y_i) = 1 \]
\[ \Rightarrow x_i \wedge y_i = 1 \] for some \( i \in \{1, 2, \ldots, n\} \).
\[ \Rightarrow x_i = 1, y_i = 1 \] for some \( i \in \{1, 2, \ldots, n\} \).
\[ \Rightarrow \underline{x} = \underline{y}. \]

4. **LINEAR COMBINATION OF VECTORS AND BASIS FOR FUZZY INNER PRODUCT SPACES**

In this part of this paper, we introduced the concept of dimension to fuzzy inner product spaces. It is based on the notion of basis.

**Definition 7. (Linear combination of vectors)** Let \( E \) be a subset of \( V_n \). A vector \( \underline{x} \in V_n \) is said to be linear combination of vectors of \( E \) if there exist a finite subset \( \{x_1, x_2, \ldots, x_m\} \) of \( E \) and \( y_1, y_2, \ldots, y_m \in \mathfrak{S} \) such that \( \underline{x} = y_1 \underline{x}_1 + y_2 \underline{x}_2 + \cdots + y_m \underline{x}_m \).

Let \( E \subseteq V_n \). The set of all linear combination of vectors of \( E \) is called the linear span of \( E \) and is denoted by \( L(E) \).

A subset \( E \) of \( V_n \) is said to be a generating subset of \( V_n \) if all vectors of \( V_n \) are linear combination of vectors of \( E \).

A subset \( E \) of \( V_n \) is said to be free if \( \sum_{i=1}^m b_i \underline{a}_i = \sum_{j=1}^k d_j \underline{c}_j \) for any \( b_i, d_j \in \mathfrak{S} - \{0\} \) and \( \underline{a}, \underline{c} \in E \) with \( i = 1, 2, \ldots, m ; j = 1, 2, \ldots, k \), then, \( m = k, \{b_1, b_2, \ldots, b_m\} = \{d_1, d_2, \ldots, d_k\} \) and \( \{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_m\} = \{\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_k\} \).

Thus, a set \( E \) is free when a linear combination of elements in \( E \) has unique non-zero coefficients and associated vectors of \( E \).

**Definition 8. (Basis)** A subset \( E \) of \( V_n \) is said to be a basis of \( V_n \) if \( E \) is generating and free i.e. every element of \( V_n \) can be written as a unique linear combination of elements of \( E \) with non-zero coefficient.
Now we observe that a basis must be made of unit vectors.

**Theorem 3.** Let $E$ be a basis of $V_n$. If $x \in E$ then $\|x\| = 1$

**Proof.** If possible let, $0 = (0,0,0,\ldots,0) \in E$.

Then $1 = (1,1,\ldots,1) \in V_n$ and $1 = 1 + 1 = 1 + 0$. Thus, $1$ can be written as two distinct linear combination of elements of $E$.

This is a contradiction because $E$ is free.

Thus, $0 \notin E$.

If possible let, $x \in E$ with $\|x\| \neq 1$. Then $x = 1x$ and $x = \|x\| \neq x$ because $\|x\| = \sqrt{x_i}$,

$$\|x\| = \sum_{i=1}^{n} (x_i \wedge x_1, x_i \wedge x_2, \ldots, x_i \wedge x_n) = (x_1, x_2, \ldots, x_n) = x.$$  

In this case, $x$ can be written as two distinct linear combinations of elements of $E$ with non-zero coefficient. This is a contradiction, because $E$ is a basis.

So, our assumption $x \in E$ with $\|x\| \neq 1$ is wrong.

Hence, $\|x\| = 1$.

The second observation is:

**Theorem 4.** Let $E$ be an orthonormal set in $V_n$. Then $E$ is free.

**Proof.** Let $x \in V_n$ with $x = \sum_{i=1}^{m} b_i a_j = \sum_{i=1}^{k} d_i c_i$, where $a_1, a_2, \ldots, a_m; c_1, c_2, \ldots, c_k \in E$ and $b_1, b_2, \ldots, b_m; d_1, d_2, \ldots, d_k \in \mathbb{S} - \{0\}$.

Now, $d_i = \langle x, c_i \rangle$ for $i = 1, 2, \ldots, k$.

If $c_j \notin \{a_1, a_2, \ldots, a_m\}$ for some $j \in \{1, 2, \ldots, k\}$, then $d_j = \langle c_j, x \rangle = \langle c_j, \sum_{i=1}^{m} b_i a_j \rangle = 0$, which is a contradiction because $d_j \in \mathbb{S} - \{0\}$.

Thus, $c_j \in \{a_1, a_2, \ldots, a_m\}$ for $j = 1, 2, \ldots, k$

$$\Rightarrow \{c_1, c_2, \ldots, c_k\} \subseteq \{a_1, a_2, \ldots, a_m\}.$$  

Also, $b_i = \langle x, a_i \rangle$ for $i = 1, 2, \ldots, m$. If $a_j \notin \{c_1, c_2, \ldots, c_k\}$ for some $j \notin \{1, 2, \ldots, k\}$, then $a_j = \langle a_j, x \rangle = \langle c_j, x \rangle = \langle a_j, \sum_{i=1}^{k} d_i c_i \rangle = 0$, which is a contradiction because $a_j \notin \mathbb{S} - \{0\}$.

Thus, $a_j \in \{c_1, c_2, \ldots, c_k\}$

$$\Rightarrow \{a_1, a_2, \ldots, a_m\} \subseteq \{c_1, c_2, \ldots, c_k\}.$$  

Therefore, $\{a_1, a_2, \ldots, a_m\} = \{c_1, c_2, \ldots, c_k\}$

Thus, $m = k$. 

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Now, \( b_j = \langle x, a_j \rangle = \langle x, e_j \rangle = d_j \) for all \( i = 1,2,\ldots,m \) and \( j \in \{1,2,\ldots,k\} \).
Hence, \( E \) is free.

**Corollary 2** A subset \( E \) of \( V_n \) is an orthonormal basis of \( V_n \) if it is orthonormal generating subset of \( V_n \).
We illustrate this result by an example:

**Example 4** Let \( \delta_1 = (1,0,0.5,0,0) \in V_4 \), \( \delta_2 = (0,0,0.5,1,0) \in V_4 \) and \( L(\delta_1,\delta_2) \) be the linear span of the vectors \( \delta_1, \delta_2 \).
Here \( \Vert \delta_1 \Vert = 1 \) and \( \Vert \delta_2 \Vert = 1 \) with \( \langle \delta_1, \delta_2 \rangle = 0 \). Thus, the set \( E = \{\delta_1, \delta_2\} \) is an orthogonal generating set.
Hence, \( E = \{\delta_1, \delta_2\} \) is an orthonormal basis for \( L(\delta_1,\delta_2) \).

### 4.1. CONSTRUCTION OF ORTHONORMAL BASIS FROM ORTHO-STOCHASTIC VECTOR

Let \( x = (x_1, x_2, \ldots, x_n) \in V_n \) be an ortho-stochastic vector.

The canonical basis or standard basis of \( V_n \) is defined as the basis \( \{e_1, e_2, \ldots, e_n\} \) with \( e_1 = (1,0,\ldots,0), e_2 = (0,1,\ldots,0), \ldots, e_n = (0,0,\ldots,1) \).
Let us construct \( \hat{\delta}_i = (x_1, x_1, x_2, x_1, x_2, \ldots, x_{i-1}) \) for all \( i = 1,2,\ldots,n \).
Then \( \Vert \hat{\delta}_i \Vert = \langle \hat{\delta}_i, \hat{\delta}_j \rangle = \sqrt{n} \langle x_i, x_i \rangle = \sqrt{n} x_i = 1 \), because \( x \) is stochastic.

Also for \( i \neq j \), we have \( \langle \hat{\delta}_i, \hat{\delta}_j \rangle = \sqrt{n} x_i x_j = 0 \) because \( x \) is ortho-vector.

Therefore \( \{\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n\} \) is an orthonormal subset of \( V_n \).
Again,
\[
\begin{align*}
e_1 & = x_1 \hat{\delta}_1 + x_2 \hat{\delta}_2 + \cdots + x_n \hat{\delta}_n \\
e_2 & = x_2 \hat{\delta}_1 + x_2 \hat{\delta}_2 + \cdots + x_n \hat{\delta}_n \\
& \quad \cdots \\
e_n & = x_n \hat{\delta}_1 + x_n \hat{\delta}_2 + \cdots + x_n \hat{\delta}_n
\end{align*}
\]
Therefore, \( L(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \subseteq L(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n) \)
\( \Rightarrow V_n \subseteq L(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n) \).
Therefore, \( V_n = L(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n) \).
Thus, \( \{\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n\} \) is an orthonormal generating subset of \( V_n \).
Hence, \( \{\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n\} \) is an orthonormal basis of \( V_n \).

**Theorem 5** Let \( a_1, a_2, \ldots, a_m \in V_n \) and \( W = L(a_1, a_2, \ldots, a_m) \). If some \( a_j \) is a linear combination of \( a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \); then \( W = L(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \)

**Proof.** Let \( E = \{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m\} \).
Let there exists \( c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_m \in \mathcal{A} \) such that
\[
a_i = c_i a_1 + c_2 a_2 + \cdots + c_{i-1} a_{i-1} + c_{i+1} a_{i+1} + \cdots + c_m a_m.
\]
Since \( \{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m\} \subseteq E \), so, \( L(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \subseteq L(E) = W \).

Let \( \overline{x} \in W \).

Then there exists \( d_1, d_2, \ldots, d_m \in \mathcal{A} \) such that
\[
x = d_1 a_1 + d_2 a_2 + \cdots + d_i a_i + \cdots + d_m a_m.
\]
\[
= d_1 a_1 + d_2 a_2 + \cdots + d_i (c_i a_1 + c_2 a_2 + \cdots + c_{i-1} a_{i-1} + c_{i+1} a_{i+1} + \cdots + c_m a_m) + \cdots + d_m a_m
\]
\[
= \sum_{j=1, j \neq i}^m (c_j d_i + d_j) a_j \in L(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)
\]

Thus, \( W \subseteq L(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \)

Hence, \( W = L(a_1, a_2, \ldots, a_m) = L(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \).

5. PROPERTIES OF LINEAR TRANSFORMATION AND ISOMETRY

In this part of the paper, we introduced the concepts and properties of linear transformation, subspaces generated by orthonormal set, ortho-stochastic subspace and isometry. The special interest to us will be isometry and its properties.

**Definition 9 (Linear transformation)** Let \( V_m \) and \( V_n \) be two fuzzy vector spaces over the fuzzy algebra \( \mathcal{A} \). A mapping \( T : V_n \mapsto V_m \) is called a linear transformation when for all \( a \in \mathcal{A} \), \( b, c \in V_n \) we have \( T(ab + c) = aT(b) + T(c) \). Here \( T(0) = 0 \) because \( T(0) = T(00) = 0T(0) = 0 \).

We call \( T \) an operator on \( V_n \) when \( T \) is linear from \( V_n \) to \( V_n \). An operator \( T \) on \( V_n \) is said to be invertible if there exist an operator \( S \) such that \( S \circ T = T \circ S = I \), where \( I : V_n \mapsto V_n \) is the linear operator \( I(x) = x \) for all \( x \in V_n \).

**Definition 10 (Subspace)** A subset \( U \subseteq V_n \) is said to be subspace of \( V_n \) if \( U \) is generated by an orthonormal set \( A = \{x_1, x_2, \ldots, x_m\} \), i.e., \( U = L(A) = \{\sum_{i=1}^m y_i x_i : y_1, y_2, \ldots, y_m \in \mathcal{A}\} \).

A subspace may not contain any ortho-stochastic orthonormal basis, for example if \( a \in \mathcal{A} \) with \( a \neq 0 \), then \( U = \{b(1, a) : b \in \mathcal{A}\} \) is a subspace of \( V_2 \) with basis \( (1, a) \). Here \( U \) does not contain any ortho-stochastic vector.

**Definition 11 (Ortho-stochastic subspace)** A subset \( U \subseteq V_n \) is said to be ortho-stochastic subspace of \( V_n \) if it has an ortho-stochastic orthonormal basis.
Example 5 Let us consider the fuzzy vector space \( V \). Let

\[
U = \{ x = (x_1, x_2, x_3) \in V : x_3 = 0, x_1, x_2 \in \mathbb{S} \}. \text{ Then } U \text{ is a subspace of } V \text{ with orthonormal basis } A = \{ (1,0,0), (0,1,0) \}, \text{ which is ortho-stochastic. Thus, } U \text{ is a ortho-stochastic subspace of } V. \\

Definition 12 (Isometry) A linear map \( T : U \rightarrow V \) between two subspaces \( U \) and \( V \) of \( n \) and \( m \) respectively, is called an isometry when for all \( x, y \in U \), we have \( \langle T(x), T(y) \rangle = \langle x, y \rangle \).

We are now ready to show the first observation, which is the core observation for isometry.

Theorem 6 Let \( T : U \rightarrow V \) be an isometry, where \( n \subseteq V_n \) and \( m \subseteq V_m \). Then \( T \) is injective.

Proof. Let \( T : U \rightarrow V \) be an isometry, where \( n \subseteq V_n \) and \( m \subseteq V_m \). Also, let \( x, y \in U \) with \( T(x) = T(y) \) and \( A = \{ \delta_1, \delta_2, \ldots, \delta_k \} \) be an orthonormal basis of \( U \).

Now, \( x_i = \langle x, \delta_i \rangle \)
= \( \langle T(x), T(\delta_i) \rangle \), because \( T \) is an isometry.
= \( \langle T(y), T(\delta_i) \rangle \), because \( T(x) = T(y) \).
= \( \langle y, \delta_i \rangle \) because \( T \) is an isometry.
= \( y_i \) for \( i = 1,2, \ldots, k \).

Therefore, \( x = y \).

Hence, \( T \) is injective.

The main result for this part is stated below.

Theorem 7 Let \( U \subseteq V_n \), \( V \subseteq V_m \) be two subspaces and \( T : U \rightarrow V \) be a linear map. Then the followings are equivalent.

(i) For any orthonormal set \( A = \{ \delta_1, \delta_2, \ldots, \delta_k \} \) of \( U \), the set \( \{ T(\delta_1), T(\delta_2), \ldots, T(\delta_k) \} \) is an orthonormal set of \( V \).
(ii) There exists an orthonormal basis \( A = \{ \delta_1, \delta_2, \ldots, \delta_k \} \) of \( U \) such that \( \{ T(\delta_1), T(\delta_2), \ldots, T(\delta_k) \} \) is an orthonormal set of \( V \).
(iii) \( T \) is an isometry.

Proof. (i) \( \Rightarrow \) (ii)
Since \( A = \{ \delta_1, \delta_2, \ldots, \delta_k \} \) is the orthonormal basis of \( U \), so, \( A \) is an orthonormal set of \( U \).
Thus, by (i) \( \{ T(\delta_1), T(\delta_2), \ldots, T(\delta_k) \} \) is an orthonormal set of \( V \).
(ii) \( \Rightarrow \) (iii)
Let \( A = \{ \delta_1, \delta_2, \ldots, \delta_k \} \) be a basis of \( U \) such that \( \{ T(\delta_1), T(\delta_2), \ldots, T(\delta_k) \} \) is an orthonormal set of \( V \).
Let \( x, y \in U \). Then \( x = \sum_{i=1}^{k} x_i \hat{\delta}_i, y = \sum_{i=1}^{k} y_i \hat{\delta}_i \) where \( x_i, y_i \in \mathbb{S} \) for \( i = 1, 2, \ldots, k \).

Now, \( \langle T(x), T(y) \rangle = \sum_{i=1}^{k} x_i T(\hat{\delta}_i), \sum_{j=1}^{k} y_j T(\hat{\delta}_j) \rangle = \sum_{i,j=1}^{k} x_i y_j \langle T(\hat{\delta}_i), T(\hat{\delta}_j) \rangle \)

\[ = \sum_{i=1}^{k} x_i y_i \langle T(\hat{\delta}_i), T(\hat{\delta}_i) \rangle, \text{ because } T(\hat{\delta}_2), \ldots, T(\hat{\delta}_k) \}\] is an orthonormal set of \( V \).

\[ = \sqrt{\sum_{i=1}^{k} x_i y_i} \]

\[ = \langle x, y \rangle. \]

Thus, \( T \) is an isometry.

(iii) \( \Rightarrow \) (i)

Let \( T \) be an isometry.

Then \( \langle T(x), T(y) \rangle = \langle x, y \rangle \) for all \( x, y \in U \).

Now, \( \langle T(\hat{\delta}_i), T(\hat{\delta}_j) \rangle = \langle \hat{\delta}_i, \hat{\delta}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \)

Hence, \( \{T(\hat{\delta}_1), T(\hat{\delta}_2), \ldots, T(\hat{\delta}_k)\} \) is an orthonormal set.

### 5.1 ISOMORPHISM AND ISOMORPHIC SUBSPACES

In the following, we study further into the isomorphism and isomorphic subspaces in order to obtain the results that an isomorphism maps orthonormal bases to orthonormal bases and any two orthonormal bases of a subspace in a fuzzy vector space have the same cardinality.

**Definition 13 (Isomorphism)** Let \( U \subseteq V_n, V \subseteq V_m \) be two subspaces and \( T : U \mapsto V \) be a linear map. Then \( T \) is said to be an isomorphism if \( T \) is a surjective isometry. In this case, \( U \) and \( V \) are called isomorphic subspaces.

It is clear that the composition of two isomorphism is also an isomorphism and the inverse of an isomorphism is an isomorphism.

**Theorem 8** Let \( U \subseteq V_n, V \subseteq V_m \) be two subspaces and \( T : U \mapsto V \) be an isomorphism. Then \( T \) maps orthonormal bases to orthonormal bases.

**Proof.** Since \( T : U \mapsto V \) be an isomorphism.

Thus, \( T \) is a surjective isometry.

Let \( \{x_1, x_2, \ldots, x_n\} \) be an orthonormal basis of \( U \).

Then \( \{T(x_1), T(x_2), \ldots, T(x_n)\} \subseteq V \).

Thus, \( L\{T(x_1), T(x_2), \ldots, T(x_n)\} \subseteq V \).

Let \( y \) be an arbitrary element of \( V \).

Since \( T \) is surjective, so, there exists \( x \in U \) such that \( T(x) = y \).
Since \( x \in U \) and \( \{x_1, x_2, \ldots, x_t\} \) is an orthonormal basis of \( U \).

Thus, there exists \( c_1, c_2, \ldots, c_t \in \mathcal{S} \) such that \( x = \sum_{i=1}^{t} c_i x_i \).

Now, \( y = T(x) = T(\sum_{i=1}^{t} c_i x_i) = \sum_{i=1}^{t} c_i T(x_i) \in L\{T(x_1), T(x_2), \ldots, T(x_t)\} \)

Therefore, \( V = \{T(x_1), T(x_2), \ldots, T(x_t)\} \).

Thus, \( \{T(x_1), T(x_2), \ldots, T(x_t)\} \) is a generating subset of \( V \).

Since \( T \) is an isometry and \( \{x_1, x_2, \ldots, x_t\} \) is an orthonormal basis.

Thus, \( \{T(x_1), T(x_2), \ldots, T(x_t)\} \) is an orthonormal generating subset of \( V \).

Hence, \( \{T(x_1), T(x_2), \ldots, T(x_t)\} \) is an orthonormal basis of \( V \).

**Theorem 9** Any two orthonormal bases of a subspace \( U \) of \( V_n \) have the same cardinality.

**Proof.** Let \( W_1 = \{x_1, x_2, \ldots, x_t\} \) and \( W_2 = \{y_1, y_2, \ldots, y_t\} \) be two orthonormal bases of \( U \).

Then there exist isomorphism \( T_1 : U \mapsto V_s \) and \( T_2 : U \mapsto V_t \).

Thus, \( T_1 \circ T_2^{-1} : V_t \mapsto V_s \) is an isomorphism.

Therefore, \( V_t \) and \( V_s \) are two isomorphic subspaces.

Hence, \( t = s \).

### 6. ORTHOGONAL COMPLEMENT OF FUZZY VECTORS AND ITS PROPERTIES

**Definition 14 (Orthogonal complement)** Let \( V_n \) be a fuzzy vector space and \( x \in V_n \). Then the orthogonal complement of \( x \) is denoted by \( x^\perp \) and is defined by \( x^\perp = \{y \in V_n : \langle x, y \rangle = 0\} \).

Let \( U \) be a subspace of \( V_n \). Then the orthogonal complement of \( U \) is denoted by \( U^\perp \) and is defined by \( U^\perp = \{y \in V_n : \langle x, y \rangle = 0 \text{ for all } x \in U\} \).

**Example 6** Let us consider the fuzzy vector space \( V_2 \) and \( x = (0.5,0) \). Then \( x^\perp = \{y = (y_1, y_2) \in V_2 : \langle x, y \rangle = 0\} \).

Now, \( \langle x, y \rangle = 0 \)

\[
\Rightarrow \sum_{i=1}^{2} x_i y_i = 0
\]

\[
\Rightarrow max\{0.5y_1,0y_2\} = 0
\]

\[
\Rightarrow 0.5y_1 = 0 \text{ and } y_2 \text{ is arbitrary.}
\]

\[
\Rightarrow y_1 = 0 \text{ and } y_2 \text{ is arbitrary.}
\]

Therefore, \( x^\perp = \{y = (0,y_2) \in V_2 : y_2 \in \mathcal{S}\} \).
Example 7 Let us consider $V_3$ and $U = \{c \underline{x} : \underline{x} = (1,0,0), c \in \mathbb{R}\}$.

Then $U^\perp = \{\underline{y} = (y_1, y_2, y_3) \in V_3 : \langle c \underline{x}, \underline{y} \rangle = 0 \}$.

Now, $\langle c \underline{x}, \underline{y} \rangle = 0$
$\Rightarrow \langle \underline{x}, \underline{y} \rangle = 0$
$\Rightarrow \sum_{i=1}^{3} x_i y_i = 0$
$\Rightarrow \max\{1y_1, 0y_2, 0y_3\} = 0$
$\Rightarrow y_1 = 0$ and $y_2, y_3 \in \mathbb{R}$ are arbitrary.

Therefore, $U^\perp = \{\underline{y} = (0, y_2, y_3) \in V_3 : y_2, y_3 \in \mathbb{R}\}$.

Theorem 10 Let $M \subseteq V_n$ and $M^\perp$ be the orthogonal complements of $M$. Then the following holds.

(i) $M^\perp$ is a subspace of $V_n$, $M \subseteq M^{\perp\perp}$ and $M \cap M^\perp = \{0\}$.

(ii) Let $N \subseteq V_n$ with $M \subseteq N$. Then $N^\perp \subseteq M^\perp$.

(iii) $\{0\}^\perp = V_n$ and $V_n^\perp = \{0\}$ and $M^\perp = M^{\perp\perp}$.

(iv) Let $M, N \subseteq V_n$. Then $(M + N)^\perp = M^\perp \cap N^\perp$, where $M + N = \{\underline{x} + \underline{y} : \underline{x} \in M, \underline{y} \in N\}$.

Proof. (i) Let $a, b \in M^\perp$ and $\alpha, \beta \in \mathbb{R}$.

Then for any $\underline{c} \in M$, we have
$\langle \alpha a + \beta b, \underline{c} \rangle = \alpha \langle a, \underline{c} \rangle + \beta \langle b, \underline{c} \rangle = \alpha 0 + \beta 0 = 0$.

This shows that $\alpha a + \beta b \in M^\perp$.

Thus, $M^\perp$ is a subspace of $V_n$.

Again, $a \in M$.

Then $\langle a, b \rangle = 0$ for all $b \in M^\perp$.
$\Rightarrow a \in (M^\perp)^\perp = M^{\perp\perp}$.

Therefore, $M \subseteq M^{\perp\perp}$.

Again, $a \in M \cap M^\perp$
$\Rightarrow a \in M$ and $a \in M^\perp$
$\Rightarrow \langle a, a \rangle = 0$
$\Rightarrow a = 0$.

Hence $M \cap M^\perp = \{0\}$.

(ii) Let $a \in N^\perp$
\[ \Rightarrow \langle a, b \rangle = 0 \text{ for all } b \in N \\
\Rightarrow \langle a, b \rangle = 0 \text{ for all } b \in M \subseteq N \\\n\Rightarrow a \in M^\perp. \]

Therefore, \( N^\perp \subseteq M^\perp. \)

(iii) \( \{0\}^\perp = \{a \in V_n : \langle 0, a \rangle = 0 \} = V_n \) because \( \langle 0, a \rangle = 0 \) for all \( a \in V_n. \)

Also, if \( a \neq 0 \) then \( \langle a, a \rangle \neq 0 \). In other words, a non-zero element of \( V_n \) cannot be orthogonal to the entire space \( V_n. \)

Hence, \( V_n^\perp = \{0\}. \)

Let \( a \in M. \)

Then \( \langle a, b \rangle = 0 \) for all \( b \in M^\perp. \)

\[ \Rightarrow a \in (M^\perp)^\perp = M^{\perp\perp}. \]

Thus, \( M \subseteq M^{\perp\perp}. \)

Changing \( M \) by \( M^\perp, \) we get

\[ M^\perp \subseteq ((M^\perp)^\perp)^\perp = M^{\perp\perp\perp}. \]

Again \( M \subseteq M^{\perp\perp}. \)

\[ \Rightarrow (M^{\perp\perp})^\perp \subseteq M^{\perp\perp}, \text{ by (ii).} \]

\[ \Rightarrow M^{\perp\perp \perp} \subseteq M^{\perp\perp}. \]

Hence \( M^{\perp\perp} = M^{\perp\perp\perp}. \)

(iv) Let \( a \in (M + N)^\perp. \)

Then \( \langle a, b \rangle = 0 \) for all \( b \in M + N. \)

Let \( b = b_1 + b_2, \) where \( b_1 \in M \) and \( b_2 \in N. \)

Now, \( \langle a, b \rangle = 0 \)

\[ \Rightarrow \langle a, b_1 + b_2 \rangle = 0 \]

\[ \Rightarrow \langle a, b_1 \rangle \vee \langle a, b_2 \rangle = 0 \]

\[ \Rightarrow \langle a, b_1 \rangle = 0 \text{ and } \langle a, b_2 \rangle = 0 \text{ for all } b_1 \in M \text{ and for all } b_2 \in N \]

\[ \Rightarrow a \in M^\perp \text{ and } a \in N^\perp \]

\[ \Rightarrow a \in M^\perp \cap N^\perp \]

Therefore, \( (M + N)^\perp \subseteq M^\perp \cap N^\perp. \)

Conversely, let \( a \in M^\perp \cap N^\perp \)

\[ \Rightarrow a \in M^\perp \text{ and } a \in N^\perp \]

\[ \Rightarrow \langle a, b \rangle = 0 \text{ for all } b \in M \text{ and } \langle a, c \rangle = 0 \text{ for all } c \in N \]

Let \( d \in M + N \)

\[ \Rightarrow d = b + c, \text{ where } b \in M \text{ and } c \in N. \]
Now, \( \langle a, d \rangle \)  
= \( \langle a, b + c \rangle \)  
= \( \langle a, b \rangle \lor \langle a, c \rangle \)  
= \( 0 \lor 0 = 0 \) for all \( d \in M + N \) .

This shows that \( a \in (M + N)^{\perp} \).

Therefore, \( M^{\perp} \cap N^{\perp} \subseteq (M + N)^{\perp} \).

Hence \( (M + N)^{\perp} = M^{\perp} \cap N^{\perp} \).

### 7. ORTHO-STOCHASTIC MATRIX AND REFLECTION

**Definition 15** Let \( A = [a_{ij}]_{n\times n} \) be a matrix on \( V_n \). Then the transpose of \( A \) is denoted by \( A^{\ast} \) and is defined by \( A^{\ast} = [a_{ji}]_{n\times n} \).

**Definition 16 (Ortho-stochastic matrix)** Let \( A = [a_{ij}]_{n\times n} \) be a matrix on \( V_n \). Then the matrix \( A \) is said to be ortho-stochastic matrix if \( A^{\ast} A \geq I \) and \( AA^{\ast} \leq I \).

A matrix \( A \) is orthogonal if \( A^{\ast} A = AA^{\ast} = I \) i.e., \( A \) and \( A^{\ast} \) both are ortho-stochastic matrix.

**Lemma 1** A matrix \( A \) is symmetric ortho-stochastic if and only if \( A \) is an orthogonal matrix of order 2.

**Proof.** Let \( A \) be symmetric and ortho-stochastic.

Then, \( A^{\ast} A \geq I \), \( AA^{\ast} \leq I \) and \( A = A^{\ast} \)

\[ \Rightarrow AA \geq I \quad \text{and} \quad AA \leq I \]

\[ \Rightarrow A^2 \geq I \quad \text{and} \quad A^2 \leq I \]

\[ \Rightarrow A^2 = I . \]

Thus, \( AA^{\ast} = A^2 = I = A^{\ast} A \), i.e., \( A \) is an orthogonal matrix of order 2.

Conversely, let \( A \) be an orthogonal matrix of order 2.

Then \( AA^{\ast} = A^{\ast} A = I \) and \( A^2 = AA = AA = I \).

This shows that \( A \) is an invertible matrix with \( A^{-1} = A^{\ast} = A \).

Hence, \( A \) is symmetric and ortho-stochastic.

**Definition 17 (Reflection)** A matrix \( A \) is said to be a reflection if it is symmetric and ortho-stochastic, i.e., an orthogonal matrix of order 2.

**Important results:** The sum and product of reflections need not be a reflection. As for example,

Let \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Then both \( A, B \) are reflection but the sum
\[
A + B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]
is not a reflection, because \((A + B)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \neq I\).

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\]
is not a reflection, because \((AB)^* = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq AB\).

**Definition 18 (Joint trace)** Let \(A_1, A_2, \ldots, A_k\) be matrices on \(V_n\) with \(A_m = [a_{ij}^m]_{n \times n}\) for \(m = 1, 2, \ldots, k\). Then the joint trace of \(A_1, A_2, \ldots, A_k\) is denoted by \(\text{tr}(A_1, A_2, \ldots, A_k)\) and is defined by

\[\text{tr}(A_1, A_2, \ldots, A_k) = \sum_{i=1}^{n} a_{i1}^1 a_{i2}^2 \cdots a_{ik}^k.\]

In particular, the trace of \(A = [a_{ij}]_{n \times n}\) is given by

\[\text{tr}(A) = \sum_{i=1}^{n} a_{ii}.\]

A vector \(b \in V_n\) is said to be an invariant vector for \(A\) if \(Ab = b\). Also, the vector \(b \in V_n\) is said to be common invariant vector of \(A_1, A_2, \ldots, A_k\) if \(A_i b = b\) for \(i = 1, 2, \ldots, k\).

**Lemma 2** Let \(A, B\) be two matrices on \(V_n\). Then

(i) \(\text{tr}(AB) = \text{tr}(BA)\).

(ii) \(\text{tr}(BAB^*) = \text{tr}(A)\) if \(B\) is an orthogonal matrix.

**Proof.** Let \(A = [a_{ij}]_{n \times n}\) and \(B = [b_{ij}]_{n \times n}\).

Now, \(\text{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}\)

\[= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}\]

\[= \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ij} = \text{tr}(BA).\]

Also, \(\text{tr}(BAB^*) = \text{tr}(B(AB^*))\)

\[= \text{tr}((AB^*)B)\text{ because } \text{tr}(AB) = \text{tr}(BA)\]

\[= \text{tr}(A(BB^*))\]

\[= \text{tr}(A)\text{, because } B\text{ is an orthogonal matrix, so, } BB^* = B^* B = I.\]

= \text{tr}(A).
Theorem 11 An ortho-stochastic matrix $A$ has an invariant ortho-stochastic vector if and only if $\text{tr}(A) = 1$.

Proof. Let $A = [a_{ij}]_{n \times n}$ be a matrix on $V_n$.

Thus, $a_{ij} \in \mathbb{S} = [0,1]$ for $i, j = 1,2,\ldots,n$.

Therefore, $a_{ij} \leq 1$ for $i, j = 1,2,\ldots,n$.

$\Rightarrow a_{ii} \leq 1$ for $i = 1,2,\ldots,n$.

$\Rightarrow \sum_{i=1}^{n} a_{ii} \leq 1$

$\Rightarrow \text{tr}(A) \leq 1$.

Let $b = (b_1,b_2,\ldots,b_n) \in V_n$ be an ortho-stochastic invariant vector of $A$ Then, $Ab = b$ and $b_ib_j = 0$ for $i \neq j$; $i, j \in \{1,2,\ldots,n\}$ and $\sum_{i=1}^{n} b_i = 1$.

Now, $Ab = b$

$\Rightarrow \sum_{j=1}^{n} a_{ij}b_j = b_i$

Multiplying both sides by $b_i$ and since $b_ib_j = 0$ for $i \neq j$.

Thus, $a_{ii}b_i = b_i$

$\Rightarrow a_{ii} \geq b_i$

$\Rightarrow \sum_{i=1}^{n} a_{ii} \geq \sum_{i=1}^{n} b_i = 1$

$\Rightarrow \text{tr}(A) \geq 1$.

Therefore, $\text{tr}(A) = 1$.

Conversely, let $\text{tr}(A) = 1$.

Thus, $\sum_{i=1}^{n} a_{ii} = 1$.

Therefore, there exist a ortho-stochastic vector $b = (b_1,b_2,\ldots,b_n) \in V_n$ such that $b_j \leq a_{ij}$.

Since $A$ is ortho-stochastic, so, $a_{ij}a_{ji} = 0$ for $i \neq j$.

Now, $b_j \leq a_{jj}$

$\Rightarrow a_{jj}b_j \leq a_{jj}a_{ij}$

$\Rightarrow a_{ij}b_j = 0$ for $i \neq j$.

Also, $(Ab)_i = \sum_{j=1}^{n} a_{ij}b_j = a_{ii}b_i = b_i$.

Thus, $Ab = b$.

Hence, $b$ is an invariant ortho-stochastic vector for $A$.

We record the following observation as well:
Corollary 3 If $A$ is ortho-stochastic matrix and $B$ is orthogonal matrix on $V_n$. Then $A$ has an invariant ortho-stochastic vector if and only if $BAB^*$ has an invariant ortho-stochastic vector.

Proof. $A$ has an invariant ortho-stochastic vector

$\iff tr(A) = 1$

$\iff tr(BAB^*) = 1$, by lemma 2.

$\iff BAB^*$ has an invariant ortho-stochastic vector.

8. CONCLUSIONS

We can use the concept of ortho-stochastic vector in Operator Theory. Also, the concepts of orthogonal complement of fuzzy vector can use to calculate the spectrum of an operator and to prove spectral theorem for compact self-adjoint operators. The results for isometry and isomorphic subspaces can use to developed the concept of partial isometry, orthogonal projection and square root of an non-negative operator. The concept presented in this paper are not limited to a specific application Thus, the results of our paper are paving the way to numerous possibilities for future research.

ACKNOWLEDGEMENTS

Financial support offered by Council of Scientific and Industrial Research, New Delhi, India (Sanction no. 09/599(0054)/2013-EMR-I) is thankfully acknowledged. Also, the authors are very grateful and would like to express their sincere thanks to the anonymous referees and Editor-in-Chief Robert Burduk for their valuable comments.

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