A POSSIBLE RESOLUTION TO HILBERT’S FIRST PROBLEM BY APPLYING CANTOR’S DIAGONAL ARGUMENT WITH CONDITIONED SUBSETS OF $\mathbb{R}$, WITH THAT OF $(\mathbb{N}, \mathbb{R})$.

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ABSTRACT

We present herein a new approach to the Continuum hypothesis CH. We will employ string conditioning, a technique that limits the range of a string over portions of its sub-domain for forming subsets $K$ of $\mathbb{R}$. We will prove that these are well defined and in fact proper subsets of $\mathbb{R}$ by making use of Cantor’s Diagonal argument in its original form to establish the cardinality of $K$ between that of $(\mathbb{N}, \mathbb{R})$ respectively.

KEYWORDS

Diagonal Argument, Continuum Hypothesis CH, Resolution to CH

1 PREFACE

The continuum hypothesis (CH) is one of and if not the most important open problems in set theory, one that is important for both mathematical and philosophical reasons. Philosophically and perhaps practically, mathematicians are divided on the matter of a resolution to CH. The uncanny persistence of the problem has led to several mainstream views surrounding its resolution. Discussions on the possibility of a resolution, notably one from the Institute of Advanced study at Princeton gives a vast account of the thoughts on the problem and a detailed summary of the progress made this far, along with what may constitute a solution see [13]. Most similar discussions express the current main sequence thoughts on the matter, and the division that exists amongst mathematicians in the views they hold with regard to a resolution, the nature of the resolution and what a resolution to CH may mean. Some of the mainstream views can be summarized as follows: Finitist-mathematicians believe that we only ever deal with the finite and as such simply put, we can’t really say much about the infinite. Pluralists naturally believe in the plurality of things, that any one of the outcomes of CH is possible. Though the efforts of both Cohen and Gödel showed the consistency of ZFC + /CH and ZFC + CH respectively, Cohen held a strong pluralist view that his demonstration that CH cannot be decided from ZFC alone, essentially resolved the matter. For Cohen’s independence results, See [11]. Contrary to this however, Gödel believed that a well justified extension to ZFC was all that was necessary in the way of deciding CH. Gödels program seemingly
the promising option forward aims to find an extension of ZFC axioms capable of
deciding CH. Gödel’s Program: Decide mathematically interesting questions independent of ZFC
in well justified extensions of ZFC. Gödel himself proposed the large cardinal axioms as a
candidate.
Shortly thereafter however, this extension still proved to be insufficient for the task of
deciding CH, as was demonstrated by the results of Levy and Solovay. The forcing tech-
niques employed by Cohen have since paved the way toward establishing many consist-
tency results such as the insufficiency of the large cardinal extension of ZFC in the way
of deciding CH, also by Levy and Solovay. See [12]. This however, was the starting
point for the work of W.H. Woodin demonstrating, (on the premise of large cardinals),
the effective failure of CH via use of a canonical model in which CH fails [9]. It is well
known that forcing cannot be used to decide CH, and for this primary reason, we explore
here some new ideas outside of the scope of forcing.
We must admit that these Ideas arose from Bit Conditioning, a form of conditioning infor-
mation stored as bits of data.

2 PREVIOUS ATTEMPTS AT A RESOLUTION TO CH

The origins of the Continuum Hypothesis (CH) can be traced back to the late 19th and
early 20th centuries when mathematicians were exploring the concept of infinity and the
cardinality of infinite sets.

The Continuum Hypothesis was formally stated by Cantor in 1878. It asserts that there
is no cardinality between the cardinality of the natural numbers and the cardinality of the
real numbers. In other words, there is no set with cardinality strictly greater than ℵ₀ and
strictly smaller than ℵ₁. He developed the theory of cardinality, which assigns a cardinal
number to each set, indicating the size or "bigness" of the set. Cantor discovered that
not all infinite sets have the same cardinality, and he introduced the concept of different
infinite sizes or "degrees of infinity." Cantor also showed that the cardinality of the natural
numbers (denoted by ℵ₀ or aleph-null) is the smallest infinite cardinality, and he conjet-
tured that the next larger cardinality is the cardinality of the real numbers (denoted by
ℵ₁ or aleph-one). This conjecture is essentially equivalent to the Continuum Hypothesis.
This initial insight was demonstrated by a diagonal argument bearing his name. The basic
idea of the argument is as follows:
Assume, for the sake of contradiction, that ℝ is countable, meaning its elements can be
listed as a sequence: r₁, r₂, r₃, r₄, ... Form a new real number x by constructing its decimal
representation in a specific way. In the ith decimal place, choose a digit different from
the ith digit of the number ri. For example, if the ith digit of ri is 3, select 7 for the ith
digit of x. The resulting number x is guaranteed to be different from every number in
the assumed countable list r₁, r₂, r₃, r₄, ... because it differs from each of them in at least
one decimal place. Thus, we have constructed a real number x that does not appear in
the assumed countable list, contradicting the assumption that ℝ is countable. The key
insight of Cantor’s diagonal argument is that by constructing a new element that differs
from each element of a given countable list, we can demonstrate the existence of elements
outside the list. This shows that there are "more" real numbers than can be enumerated in
a countable manner. For elementary texts on this topic, see for instance [1], [22].

Cantor made significant efforts to prove or disprove the Continuum Hypothesis but was unable to settle the question. He corresponded with fellow mathematicians and worked on the problem for many years, but a definitive answer eluded him. The Continuum Hypothesis remained a major open question in set theory until the early 20th century. In 1900, David Hilbert included the problem of the Continuum Hypothesis as one of his famous 23 unsolved problems presented at the International Congress of Mathematicians in Paris. This helped elevate the status of the problem and spurred further investigations. For a full detailed account of this see for instance [15]. The search for a resolution to the Continuum Hypothesis continued throughout the 20th century, with numerous mathematicians attempting to prove or disprove it. Notable mathematicians such as Kurt Gödel, Paul Cohen, and Saharon Shelah made significant contributions to the study of the Continuum Hypothesis and its independence from the standard axioms of set theory.

Gödel, famous for the independence results, specifically the Gödel’s Incompleteness Theorems which demonstrate the existence of undecidable propositions in Peano Arithmetic (PA), is a famous result in mathematical logic. Before we outline Gödel’s ideas on CH, we will outline his brilliant proof of the Incompleteness Theorems. At a high level, Gödel’s proof begins by representing the syntax and semantics of PA within the system itself. This encoding allows the system to reason about its own statements and proofs. Gödel constructs a method to encode formulas and proofs of PA as numbers. This encoding enables the system to manipulate and reason about its own syntactic objects. Gödel assigns unique numbers (Gödel numbers) to formulas and proofs in PA. This encoding is recursive and captures the structure of formulas and proofs. Gödel uses a diagonalization argument (Diagonal argument as is more conventional), to construct a formula that asserts its own unprovability within PA. This formula is referred to as the Gödel sentence or the diagonal lemma. By establishing the unprovability of the Gödel sentence within PA, Gödel demonstrates that there exists a true statement that is not provable in the system. This shows the incompleteness of PA. Gödel’s proof shows that any consistent formal system that can encode arithmetic, such as PA, will have undecidable propositions. These undecidable propositions cannot be proven or disproven within the system itself. See for instance [22], and [23].

To provide an outline of this fascinating result, we take from Smulliyan: Let $X$ be some encoded-expression, then the following is possible:

Let $P$ stand for printable,
$N$ norm of,
and $!$ not.

$P(X) \rightarrow True$ if $X$ is ’printable’.

$P(N(X)) \rightarrow True$ if $N(X)$ is ’printable’.

$!P(X) \rightarrow True$ if $X$ is NOT ’printable’.

$!P(N(X)) \rightarrow True$ if $N(X)$ is NOT ’printable’.

Given that ’the machine’ never prints false sentences:
The sentence $\text{PN}(!\text{PN}(X))$ is true if the norm of $(!\text{PN}(X))$ is printable, as $\text{PN}(..)$ means 'Printable, Norm of that which lies within (..)'. But this means that if we place $!\text{PN}$ within, the statement then translates to 'Printable, Norm of that which lies within (Norm of this not Printable(X))'. This either means that: the sentence is true and not printable, or it is printable and not true. The latter violates our hypothesis that the machine is only capable of printing true statements.

The significance of this is that all systems 'morphic' to the above in a manner of setting up statements, then Gödel's argument is made. The infinitely more significant result is that Arithmetic is one such formal system.

Gödel’s proof of the independence of CH builds upon this earlier work on incompleteness theorems. His ideas were highly inspired by the work of Cantor, as was the case for Turing as well. This was a truly revolutionary period of mathematical enlightenment. Gödel established that within any consistent formal system that is sufficiently 'powerful' to express arithmetic, there are true statements that cannot be proven within that system. Gödel used a technique called the constructible universe, denoted by L, which is a particular model of set theory. In this model, sets are constructed in a step-by-step fashion using a hierarchy of stages. Gödel then introduced a hierarchy of sets called the constructible hierarchy. Each stage of this hierarchy represents a level of the cumulative hierarchy of sets, and it is constructed based on the previous stages. A notion of constructible sets is then defined within his constructible hierarchy. These sets are built using formulas of set theory, and each constructible set is associated with a particular formula. The reflection principle ensures that if a statement is true at one stage of the constructible hierarchy, then it continues to be true at later stages. Gödel then showed that within the constructible universe L, the continuum hypothesis holds. In other words, within L, it is true that there is no set whose cardinality is strictly between that of the natural numbers and the real numbers. Finally, Gödel constructs a different model of set theory (referred to as the "Gödel model") in which CH is false. This model is obtained by considering a larger universe of sets that extends beyond L and introducing certain additional sets that violate CH.

In 1963, Paul Cohen presented his groundbreaking proof that the Continuum Hypothesis is independent of the standard axioms of set theory. This meant that the hypothesis cannot be proved or disproved within the existing framework of set theory. The results of Cohen were inspired by those of Gödel and Cantor.

The independence of the Continuum Hypothesis had a profound impact on the field of set theory and the understanding of mathematical infinity. It highlighted the inherent complexity and richness of infinite set theory and paved the way for further investigations into different cardinalities and the structure of the continuum. For a detailed account of independence results see for instance, [3], [4]. We here give a small account of the work done by Cohen on the Continuum Hypothesis.

First, let us define some notation. For any countable ordinal $\alpha$, let $2^\alpha$ denote the set of all functions from $\alpha$ to 2, and let $2^{<\alpha}$ denote the set of all finite functions from $\alpha$ to 2. We can think of $2^{<\alpha}$ as the set of "partial" functions from $\alpha$ to 2, i.e., functions that are only defined on a finite initial segment of $\alpha$. We order $2^{<\alpha}$ by extension, so $p \leq q$ means that $p$ extends $q$, i.e., $p$ is a stronger condition than $q$. We say that $p$ and $q$ are compatible.
(written \( p \vdash q \)) if there exists \( r \) such that \( r \leq p \) and \( r \leq q \).

Now, let \( V \) be a model of ZFC, and let \( G \) be a generic filter over \( V \) for the forcing notion \( (2^{<\alpha}, \leq) \). We say that \( G \) is a Cohen generic filter if it has the following two properties:

\( G \) is downward-closed: if \( p \in G \) and \( p \leq q \), then \( q \in G \). \( G \) intersects every maximal antichain in \( 2^{<\alpha} \), i.e., every collection \( A \) of pairwise incompatible elements of \( 2^{<\alpha} \) has a common extension in \( G \). Note that property (2) implies that \( G \) is maximal with respect to the ordering \( \leq \). In other words, if \( p \notin G \), then there exists a \( q \) such that \( q \leq p \) and \( q \) is incompatible with every element of \( G \).

We can now define the Cohen generic extension \( V[G] \) of \( V \). The universe \( V[G] \) consists of all sets that can be constructed using elements of \( V \) and elements of \( G \). Specifically, for each name \( \tau \) in \( V \), we define its interpretation \( \tau^G \) in \( V[G] \) as follows:

If \( \tau \) is a ground set, then \( \tau^G = \tau \). If \( \tau \) is a name for an element of \( 2^{<\omega} \), then \( \tau^G \) is the function in \( 2^\omega \) defined by \( \tau^G(n) = 1 \) if and only if \( m < n : \tau(m) = 1 \in G \). The key fact about Cohen forcing is that it adds a new subset of \( \omega \) to \( V \). Specifically, the set \( n \in \omega : \tau^G(n) = 1 \) is a new subset of \( \omega \) that is not in \( V \). This new subset has the property that it is not constructible from any set in \( V \). In particular, it is not constructible from any countable sequence of sets in \( V \).

To see why this is the case, suppose for contradiction that there exists a sequence \( (S_n : n < \omega) \) of sets in \( V \) such that \( n \in \omega : \tau^G(n) = 1 = \bigcup_n S_n \). Then each \( S_n \) is constructible from a countable sequence of sets in \( V \), say \( (T_{n,m} : m < \omega) \). Since \( V \) is a model of ZFC, there exists a formula \( \phi(x) \) such that for each \( n \), the set \( m < \omega : T_{n,m} \in x \) is the \( n \)-th element of the sequence \( S_n \) if \( \phi(x) \) is true, and the empty set otherwise. Since the sequence \( (S_n : n < \omega) \) is not in \( V \), there exists a Cohen condition \( p \) such that \( p \) forces \( \neg \phi(G) \). But this contradicts the fact that \( G \) intersects every maximal antichain in \( 2^{<\omega} \).

Finally, it is a well-known result that the addition of a Cohen subset of \( \omega \) to \( V \) is independent of ZFC (See [6], [7], [8]).

One way to visualize the Cohen forcing notion is to imagine a binary tree whose nodes correspond to partial functions from \( \omega \) to 2. The root of the tree corresponds to the empty function \( \{\} \), and the children of a node corresponding to a partial function \( f \) are obtained by extending \( f \) with a new pair \((n, b)\), where \( n \) is a natural number not already in \( \text{dom}(f) \) and \( b \) is either 0 or 1. The nodes are formed in a manner that have a chain for each ordinal in the base model.

At each level of the tree, we have a finite number of choices to make, corresponding to the possible values of the next unused natural number and the next bit in the binary representation of the function. At the limit levels of the tree, we have a branch for each possible function from \( \omega \) to 2. The partial order on \( P \) is defined by saying that a node corresponding to a partial function \( f \) is less than or equal to a node corresponding to a partial function \( g \) if and only if \( g \) extends \( f \), that is, \( \text{dom}(f) \) is a subset of \( \text{dom}(g) \) and \( g(x) = f(x) \) for all \( x \) in \( \text{dom}(f) \).

A generic filter for the Cohen forcing notion can be thought of as a path through the tree that includes all the branches that correspond to a condition in the filter. Intuitively, a generic filter "chooses" one branch from each level of the tree in a way that is consistent with the ordering relation.

In this way, the Cohen forcing notion allows us to construct a model of set theory in which we "choose" a particular path through the tree, corresponding to a random subset of
Forcing, as a method in set theory, allows us to construct mathematical models (forcing extensions) in which certain statements are either true or false. However, forcing cannot definitively resolve the Continuum Hypothesis (CH) because it does not provide a conclusive answer as to whether CH is true or false in the standard set-theoretic universe. The main reason forcing cannot settle CH is that it does not add any new information about the truth value of CH in the original set-theoretic universe. Instead, forcing allows us to construct additional models of set theory, called forcing extensions, in which we have more freedom to manipulate certain properties and values.

When applying forcing to the Continuum Hypothesis, we can construct forcing extensions in which CH is true and others in which CH is false. This shows that CH is independent of the standard axioms of set theory because both possibilities can be consistently realized. In other words, forcing demonstrates that there are models of set theory in which CH is true and models in which CH is false. This independence result implies that CH cannot be settled within the confines of the standard axioms of set theory alone. It indicates that additional axioms or principles beyond the standard ones are needed to establish the truth or falsity of CH (Again, see [3], [4]).

For these reasons, we will be turning to some new practical techniques that arose from a study into partial bit encryption. This study is strongly coupled with Information The-
ory which for the interested reader is a branch of mathematics and computer science that deals with the quantification, storage, and communication of information. As this is not the topic of this article, we leave some references for the interested reader. See for instance [17], [18], [19] and [20].

3 Introduction to the Main Idea

Let us pause at this juncture to take account of the intent behind the next few paragraphs, which is firstly to formulate a language capable of dealing with forming well defined subsets of $F := \bigcup_{i} f_i : \omega_{<} \to \{0, 1\}$, ($\omega_{<}$ is to mean a finite set of ordinals/ordered-numbers), by restricting the range of the functions $f_i$ to 0 over arbitrary domain values of $f_i$ in $\text{dom}(f_i) := \omega_{<} := \{0, 1, 2, \ldots, n\}$. Such functions ultimately constitute elements of the sets we are interested in. Trivial but important to note is that, set of all sequences of length $n$ having 'conditioned segments' is an example of a subset of $F$. Secondly, we aim to formulate a means of establishing cardinality of such conditioned sequence sets, now however of non-finite length, with that of $\text{SEQ} := \bigcup_{i} f_i : \mathbb{N} \to \{0, 1\}$ as a whole.

To achieve this result, we will make use of inductive arguments.

The fuzziness in the first point is best made clear with an example. Suppose we wished to form a finite set of elements of the form $\{120005, 340004, 710004, \ldots\}$, here suggestive that each element is a 6-digit-sequence in $\{0, \ldots, 5\} \to \{0, \ldots, 9\}$ with $s_2$ to $s_4$ included, being always 0. Such sequences are a subset of say all sequences of length 6.

A simplistic language, in the mathematical sense, capable of forming precisely such conditioned sequence sets will be valuable for establishing cardinality.

Returning to point two, the core diagonal argument works when one can show that for each element in some set $S$, there are infinitely many more in say $\mathbb{R}_2$ by the argument $\text{Diag}(S, \mathbb{R}_2)$ ($\mathbb{N}_2, \mathbb{R}_2$, naturals and reals in base two). A subtle nuance is that $\mathbb{N}_2 \to S$ does not need to be onto for the $\text{Diag}$ argument to work. As trivial as this sounds, this is important, for it tells us that if we are to search for a set $S$ that exists cardinally between $\mathbb{N}_2, \mathbb{R}_2$ we can be certain that $S \subset \mathbb{R}_2$ and need not be onto $\mathbb{N}_2$, for the $\text{Diag}$ argument to work.

If we consider the nature of how the diagonal argument is conventionally used, one observes that $\mathbb{R}$ is treated as comprising of all infinite sequences of the form $\mathbb{N} \to \{0, 1\}$ (See for instance Chapter 5 in [2]). In fact, every element of $\mathbb{R}$ is representable as some infinite binary sequence $\mathbb{R}_2$. This almost begs the question, can one form subsets of $\text{SEQ}$?

With some imagination, a few ideas come to mind, along the lines of those described at the beginning of this section. Perhaps by restricting Range values of $f_i$ to 0 over segments of its domain. Surely, all elements of the form $S := \bigcup M_10M_200M_3000M_40000\ldots$ with $M_i := \{f_i : \{0, \ldots, e\} \to \{0, 1\}\} e \in \mathbb{N}$, forms a subset of $\text{SEQ}$.

This is interesting, as what then is the cardinality of $S$ with regards to $\mathbb{N}, \mathbb{R}$ respectively? What of the diagonal argument can we augment and or reuse to felicitate this comparison?
These are some of the questions that we aim to answer in this letter for consideration by the experts.

Important point for consideration is the containment of \(S\) strictly outside of \(\mathbb{N}, \mathbb{Q}\), but we will get to this later.

With our conditioned sets \(S\), as we term temporarily, we wish to achieve the following:

\[ \text{Diag}(\mathbb{N}_2, S) \quad \text{and} \quad \text{Diag}(S, \mathbb{R}_2), \]

using \text{Diag} in short-form to denote Cantor’s diagonal argument, between the sets within brackets (Such as for the well established one between \((\mathbb{N}, \mathbb{R})\)). One would then have to make a case for using the diagonal argument interchangeably in the following sentences (Why this is so will become clear later on, and is the main focus of this article).

A) Given \(0, 1, 2, ..., 12, ..., 1000, ...\) (i.e., all naturals) there are infinitely many more Reals
B) Given some arbitrary collection \(e_1, ...., e_n, .... \subset \mathbb{R}\) there are infinitely many more Reals
C) Given some arbitrary collection \(1, 2, ...., 12, ..., 1000, ..., (i.e., \text{all naturals})\), there are infinitely many more \(e_1, ...., e_n, .... \in S \subset \mathbb{R}\)

All of which should be establish-able via \text{Diag}(\mathbb{N}, S) \quad \text{and} \quad \text{Diag}(S, \mathbb{R}).

The challenge would then be to pick such a collection, i.e. one possessing the characteristics necessary to enable \text{Diag}(\mathbb{N}, S) \quad \text{and} \quad \text{Diag}(S, \mathbb{R}).

Fortunately, this can be any collection \(S\).

4 Conceptual Ideas and Formulation

The aim of this section is to provide some definitions used for forming well defined sub-sets of \(\bigcup f_i : \omega_\prec \to \{0, 1\}\) having the properties expressed previously. We want to ultimately make an inductive argument involving finite sequences that will necessarily have to extend to infinite ones.

**Definition** (Sequence-Function)

We define a \textbf{finite} sequence-function to be a function \(f : \omega_\prec \to \{0, 1\}\), having \(\omega_\prec\) the ordered set of numbers starting 1 (strictly for ease), up to some arbitrary \(n \in \mathbb{N}, \{1, ..., n\}\), as its domain, and \(\text{range}(f) := \{1, 0\}\).

Such functions, and sets thereof, are denoted by the symbols \(f, s, S, S'\) respectively and throughout this article unless otherwise specified.

**Definition** (Sequence-Function Sets)

We define any set of the form \(S := \bigcup f_i : \omega_\prec \to \{0, 1\}\), to be a sequence-function set,
typically denoted \( S, S' \).

**Definition (Sum)**

Given an arbitrary sequence-function \( f \), the binary operation \( +_A \) on \( f \in S \), is defined as \( f +_A f = Bnum^{-1}(Bnum(f) + Bnum(f)) \), where \( Bnum(f) \) is the function \( Bnum(f) \mapsto b \in \mathbb{B} \) with \( b \) being the binary number equivalent of the image of \( f \), \( \mathbb{B} \) the binary number set as we have labelled it, \( \{+\} \) being the standard arithmetic addition operation, and \( Bnum^{-1} \) being the function \( Bnum^{-1}(b \in \mathbb{B}) \mapsto f \) with \( b \) being the binary equivalent of the ordered Image(f).

**Remarks**

\( Bnum(f) + Bnum(f) \) is written \( A_2 \), and in general \( A_j \) for many such sums.

As an example: \( A_2(1001) = 10010 \). It is to be clearly mentioned that the domain of the function(singular) remains unaffected after Summing which strictly affects the image of \( f \) alone.

**Definition (Conditioned sub-sequence)**

Given an arbitrary sequence-function \( f \), a segment \( I_K \) of the Image(f) over which Range(f) := \{0\}, for two or more domain values, is defined to be a conditioned sub-sequence.

**Remarks**

These are strictly partial functions \( f_K \) associated with \( f \). Both have identical domain values over \( I_K \). \( D(I_i), D(/I_i) \) respectively denote the domain sets associated with these image intervals. Here the set \( D(/I_i) \) of domain values, are associated with all domain intervals not associated with \( f_K \).

**Definition (Length)**

Given an arbitrary sequence-function, a segment-length (just length where there is no confusion) of a random image segment \( s_K := s[n_1, n_2] \in s \) of \( s \), with \( \text{Dom}(s_K) = [n_1, n_2] \), the domain interval of the the segment \( s_K \), of Image(\( f_k \)), denoted by \( \mathcal{M}(s_K[n_1, n_2]) \), \( \mathcal{M}(s_k) \) (where there is no confusion), and is defined to mean \( |n_2 - n_1| \) for \( n_i \in \mathbb{N} \).

**Remarks**

Where the Length of the entire function is concerned, we simply write \( \mathcal{M}(f) \)

To enable the choosing of such elements for \( S \subset \mathbb{R} \), we focus our attention here on sequences that are incapable of being reduced to a natural number via finite Sum. This is tricky, as we will see shortly.

It is a well known fact that certain rational numbers such as \( 1/3 = 0.3333... \) are asso-
ciated with fractional portions having sequence-like characteristics, i.e., non terminating
fractions. So naturally, if we are to talk solely about the irrational-sequences, we need
some means of removing these from the power set $2^\mathbb{N}$. Intuitively one way of attempting
this is to have some idea of what may constitute an 'irrational sequence'. It turns out that
binary sequences following a certain schema (SH) belong (Or are comparable to) to a sub-
set SS, of the irrational numbers. Numbers such as $(Y.11111111...), (Y.333333...)$ having a
sequence-like fractional portion, i.e., an infinite sequence of numbers, can be transformed
into to a natural number via finite Sum, thus In searching for such a schema, it should be
noted that such periodically repetitive sequences are to be discounted.
Formulation of the schema (SH)(i.e., A generalization of sorts of types of arrangements
of 0's and 1's possible, forming binary sequences), holds a strong relationship with the
effect addition has on binary numbers. Consider the simple case of $010010 + 010010 =
100100$

The alignment of (1) symbols in such summations, when added, results in the position
of (1) shifting a position to the left. 0's in alignment have no bearing on the result aside
from its sum resulting in 0. The formulation of $SH$ thus requires that the binary sequences
associated with irrational-sequences are so arranged, that it is impossible to form, via finite
additions of some $s$ to itself, $A_j(s) = X.111...$ for $j, X \in \mathbb{N}$.

(SH) begs an alternate schema to those that are periodically recursive, such as for instance :$((110001100011000...)$. Such recursive schemas hold the property that via finite Sum result
in $111111...$

As an example: $(110001100011000...) +_A ... +_A (1110011110111001... = (111111111111111111...$

It is the above consideration that sparked the idea that targeting the number of 0's between
pairs of 1's forming a binary sequence is what holds the key to forming (SH) i.e., a schema
not having this property). Noteworthy is the observation that a means of forming a non-
recursive binary sequence is by increasing the number 0's between pairs of 1's, and as
a natural extension to this is having arbitrary finite-length sequences $M_i$ spaced suchlike
forming the sequence.

It is almost arbitrary why such sequences would form part of the irrational sequences, as,
conditioned sub-sequences larger in length so to say, require more Summing in the way
of resulting in $1111...$ spanning its length. If there is always in existence one such condi-
tioned sub-sequence greater in length than all preceding conditioned sub-sequences, then
no amount of Sum on such a sequence is sufficient in the way of resulting in $111...$

**Definition** (Unconditioned sub-sequence)

Given an arbitrary sequence-function $f$ of arbitrary length, an unconditioned sub-sequence
$P_i$ of $f$, is defined to mean a segment of $\text{Image}(f)$ where $\text{Range}(f) := \{0, 1\}$.

**Remarks**

A formidable task in set theory is precision in defining sets. Whereof we do not know,
thereof we must remain silent. Present theory struggles in the ability to form precisely a col-
clection of $\bigcup f_i : \mathbb{N} \to \{0, 1\}$, primarily because it requires the existence of a choice function
and the acceptance of ZFC-axioms of arithmetic. As its existence is highly debated we
need to be precise and relay caution. What we wish to do is condition the functions both
inductively and in a precisely defined manner, allowing for set formation. In order to achieve this, we will need to make use of these definitions.

**Definition (Imposed Sequence Sets)**

Let $I = \bigcup_{i} D(I_i)$ be a union/collection of non-overlapping conditioned sub-sequence domain-intervals of conditioned sub-sequences, of an arbitrary sequence-function $f$ of sufficient length. An *imposed* sequence-function is defined to be any arbitrary sequence-function $f : \omega \rightarrow \{0,1\}$ with the conditioning:

$$s = \left\{ \begin{array}{ll}
\text{Range}(f) := \{0,1\} & \text{if } d \in \text{dom}(f), d \in \bigcap_{i} D(I_i) \\
\text{Range}(f) := \{0\} & \text{if } d \in \text{dom}(f), d \in \bigcap_{i} D(I_i) \end{array} \right.$$

written $f \leftarrow (I_i)$. Should the entire set $S$ abide by the conditioning, we then write $S \leftarrow (I_i)$ and $S$ is said to be imposed by $I_i$, so long as there is no confusion in the meaning.

**Remarks**

We only ever require conditioned sub-sequences to define an *imposition*, however we do provide additional information in certain circumstances to effect clarity by including the non-segment-interval domain values in the arguments that follow.

**Definition (Unconditioned sub-sequence Combination $C_i(g)$)**

The set $C_i(g)$ is defined to be the set of functions $\bigcup_{i} f_i : \{1,\ldots,g\} \rightarrow \{0,1\}$, $g \in \mathbb{N}$.

**Definition (Span Sets, we denote as span($S \leftarrow (I_i,P_i)$), (Simply span($I_i,P_i)$))**

Given an arbitrary sequence-function $f$ of arbitrary length, $\text{span}(f)$ for $f \leftarrow (I_i,P_i)$, $\mathcal{M}(P_i), \mathcal{M}(I_i) \neq 0\forall i$, of arbitrary Conditioned and Unconditioned sub-sequences forming $f$, is defined to be the set of all functions $\bigcup_{j} f_j \leftarrow (I_i,P_i)$.

**Definition (Sequence-Function Set Product)**

Let $S, S', U$ be arbitrary sequence-function sets such that $\Pi := S \cup S' \cup U$, and let $\forall i, j, f_i \in S$ and $f_j' \in S'$. The multiplication of $(S, S')$ is defined to be the binary operation $\Pi \times \Pi \rightarrow \Pi : S \otimes S' : \bigcup_{i,j} \{f_i +_A f_j'\} \in U$.

Such sets are comparable to a direct product set.

**Written** $S \otimes S'$, **again**, where $S \otimes S' = U : \{f_1 +_A f_1', f_1 +_A f_2', \ldots, f_1 +_A f_j', f_2 +_A f_1', f_2 +_A f_2', \ldots, f_2 +_A f_j', \ldots, f_i +_A f_1', f_i +_A f_2', \ldots, f_i +_A f_j'\}$

**Remarks**
A natural implication of span multiplication is a change in the set of conditioned sub-sequences and unconditioned sub-sequences imposed on the resulting set.

Figure 2: The above figure illustrates a series of arbitrary elements belonging to \( \text{span}(S) \) where \( S \leftarrow (I_i, P_i) \). Units within the grey area are indicative of the unconditioned sub-sequences associated with each sequence depicted.

**Definition (Set of Irrational Sequences)**

Any set \( SR \) of all non-duplicate sequence-functions \( s \leftarrow (I_i, P_i) \) having the property \( \mathcal{M}(I_i) > \mathcal{M}(I_{i-1}), \forall i \in \mathbb{N} \), is defined to be a set of irrational sequences.

**Definition \( (1_{q_1,t_1}) \)**

Given an arbitrary sequence-function \( f \in S \), \( 1_{q_1,t_1} \) is defined to be an unconditioned-segment of \( f \), \( P(f)[q_1,t_1] \), with 1’s spanning its length.

**Lemma 0**

\( \forall s \in SR, \text{ no } j \in \mathbb{N} \text{ exists such that } A_j(s) = 111.... \)

**proof**

Let \( I_{q_1}[q_1,t_1], I_{q_2}[q_2,t_2] \) be arbitrary consecutive conditioned sub-sequences of \( s \in SR \), if \( A_j(S) \) results in \( P_{q_1}[q_1,t_1] = 1_{q_1,t_1} \), then \( A_{g}(S)|g > j \) is required in order for \( P_{q_2}[q_2,t_2] \) to result in \( 1_{q_2,t_2} \). Since the chosen conditioned sub-sequences in concern are arbitrary, for infinite sequence-functions imposed in this manner, no finite set of Sums exists such that \( A_{g}(S) = 11111.... \).

We can be certain that any sequence following an *irrational* progression will not be reducible to 0.1111..., and this is the exact property we require to keep such sequences from exhibiting properties of \( \mathbb{Q} \) when unrestricted in length, in a manner of speaking.

**Definition (Conditioned sub-sequence Removed Sequence Set)**
Given an arbitrary sequence-function \( s \in S \) of some sequence-function set \( S \) having conditioned sub-sequences and unconditioned sub-sequences \((I_i, P_i)\) respectively. The sequence-function \( s' \) having the ordered arrangement of \( P_i | \forall i \) as its image is defined to be a conditioned sub-sequence removed sequence-function.

The set of all such sequence-functions of \( S \) is defined to be the conditioned sub-sequence removed sequence set associated with \( S \), written \( S/I \).

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**Definition (Half Paired Elements)**

Given an arbitrary sequence function \( s \leftarrow (I_i, P_i) \). The resulting function formed by conditioning the first half of each unconditioned sub-sequence \( P_i | \forall i \) of \( s \) with a conditioned sub-sequence (sub-sequence conditioning) of length \( \mathcal{M}(P_i)(\text{Mod}_L)2 \), is defined to be a half paired function element \( /s \) of \( s \).

**Definition (Half Conditioned sub-sequence Extended Sequence Set)**

A half conditioned sub-sequence extended sequence set is defined to be the set \( /S \) of all non-duplicate half-paired function elements associated with a set \( S \) of sequences.

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5 **The Theory**

We will devote the second part of this article toward establishing the cardinality of \( S_R \) with respect to \( N \) and \( R \) respectively. We will achieve this by forming a ratio of the form: \( |(N)| : |(R)| \) as \( |(N)| : |(H)| \), with \( H \subset S_R \subset R \) and for \( H \subset S_R \), \( |(H)| : |(R)| \) as \( |(N)| : |(R)| \). (\( S_R \) here refers to the collection of all Irrational-Sequences of indefinite length). The difficulty in establishing this result surrounds \( H \) being a subset of \( S_R \) and as such any attempt at pairing elements of \( H \) with those of \( R \) becomes challenging for obvious reasons. We will however formulate a means of overcoming this challenge by implementing a technique of pairing and shrinking via denotation, that which is in one-to-one correspondence.
with the sequence set $H$. Finally we will argue that $\mathbb{R}$ has a cardinally larger spanning-set than $H$ by demonstrating that the unpaired remainder $K \subset S_{\mathbb{R}}$ is such that any attempt at pairing elements of $K \rightarrow \mathbb{R}$ respectively is much the same as attempting the pairing $(\mathbb{N}, \mathbb{R})$.

Before continuing, we prove the following important lemma.

**Lemma 1.0**

If a sequence-function $s$ is formed having an image mapped to the diagonal values $d$ of a set of sequence-functions $s_1, ..., s_n \in D \leftarrow (I_i, P_i)$, then $s \leftarrow (I_i, P_i)$ and as such constitutes an element of $D$.

**Proof**

Any conditioned sub-sequence of length $k$, makes use of $k$ elements and thus $k$-diagonal entries in the way of producing a sequence of length $k$. For any such set of conditioned sub-sequences in alignment, the result is clearly a conditioned sub-sequence of same length, and the same is thus true of any arbitrary series of unconditioned sub-sequences in alignment as well.

### 6 Main Argument

Let all $s \in SR$ be such that $s \leftarrow (I_i, P_i)$, where $\forall i, \mathcal{M}(I_{i+1}) = \mathcal{M}(I_i) + 1$. $\forall i$, and $\mathcal{M}(P_i) = g|g \in 2n|n \in \mathbb{N}$. Let $I_i', P_i'$ be the conditioned $H$s and unconditioned sub-sequences respectively of all sequences $s' \in S'$ belonging to $s' \in hs(SR) = span(I_i', P_i')$. Let in addition $\mathcal{M}(P_i^1) + \mathcal{M}(P_i^2) = g$, with $\mathcal{M}(P_i^1) = \frac{g}{2}$, then

$$SR = span(I_i, P_i^1) \otimes span(I_i', P_i')$$

(1)

Rewriting elements of $(\Gamma_1 := span(I_i, P_i^1), \Gamma_2 := span(I_i', P_i'))$ as $\alpha_i \in \Gamma_1$, $\alpha_i' \in \Gamma_2$ respectively, we can see from (2) that $SR := \Gamma_2 \otimes span(I_i, P_i^1)$.

Now, if we try and pair $\bigcup_i \alpha_i$ with $\bigcup_i \alpha_i \otimes span(I_i', P_i') := \bigcup_i \alpha_i \otimes \bigcup_i \alpha_i'$, we note...
that the elements of both sequence-sets in one-to-one correspondence have already been *shrunk* (To shrink via denotation is to mean the re-representation of a sub-sequence via the use of a variable.), any attempt at pairing $\bigcup_{i} \alpha_{i}$ with $span(I'_{i}, P^{2}_{i})$ and $\bigcup_{i} \alpha_{i}$ is the same as attempting to pair elements of $(\mathbb{N}, \mathbb{R})$ respectively. This can be seen if one uses $span(I'_{i}, P^{2}_{i})$ in the diagonal argument instead of $\mathbb{R}$ and takes into account Lemma 1.0 in the following way:
For any arbitrary set of paired-elements $(\alpha_{1} : \alpha_{1}), (\alpha_{2} : \alpha_{2}), ..., (\alpha_{n} : \alpha_{n})$, attempting the mapping of elements of $s_{1}, s_{2}, ... \in span(I'_{i}, P^{2}_{i})$ alongside such a pairing, having in mind that $\bigcup_{i} \alpha_{i} \rightarrow \bigcup_{i} \alpha_{i}$ is obviously onto, quickly shows that one can easily find an unpaired element $s_{j} \in span(I'_{i}, P^{2}_{i})$ by employing the diagonal argument using $s_{1}, s_{2}, ... \in span(I'_{i}, P^{2}_{i})$.

$\alpha_{1} : (\alpha_{1}, 100100101111...),$
$\alpha_{2} : (\alpha_{2}, 001010100001...),$
$...$
$\alpha_{n} : (\alpha_{n}, 010101011110...)$

(Example) An attempt at pairing $\Gamma_{1} \rightarrow SR$.

7 Conclusion

As $\omega_{\infty}$ is arbitrary, the previous arguments made, can be applied inductively for all $\omega_{j}$. The results as such, follow in general as outlined in the introduction to the previous section.
One can apply the diagonal argument involving $(\mathbb{N}_{2}, SR)$. Since $SR \subset \mathbb{R}_{2}$ as already established. This is all that needs proving in the way of establishing the cardinality of $SR$.
between that of \((\mathbb{N}_2, \mathbb{R}_2)\)

References


Figure 6: Elements and Half Paired Elements.


