Estimation of mean and its function using asymmetric loss function

BinodKumar Singh

University of Petroleum & Energy Studies, Dehradun, Uttrakhand, India.
singhbinod4@yahoo.co.in, bksingh_ism@yahoo.co.in

Abstract

In this paper suggested an improve estimator for mean using Linex loss function and shows that the improved estimator dominates the Searls (1964) estimator underLinex loss function. The sufficient statistics can be used to find the uniformly minimum risk unbiased estimators. In this paper an improve estimation for μ^2 *is suggested (which uses coefficient of variation) under Linex loss function. The mathematical expression of improve estimator of fourth power of mean is also obtained and an improve estimator for common mean in negative exponential distribution is also proposed under Linex loss function.Pandey and Malik (1994) considered the estimator* $T_1' = w_1\bar{x}^2 + w_2\bar{y}^2 + w_3\bar{x}\,\bar{y}$ *for common mean with the restriction* $w_1 + w_2 + w_3 = 1$. *Here considered the above estimator for* $w_1 + w_2 + w_3 \neq 1$ *and studied its property under Linex loss function. In this paper alsoconsidered the displaced exponential distribution under Linex loss function and suggested an improve estimator.*

Key Words

Linex loss Function, Mean square error and risk

1.Introduction

Let x_1, x_2, ..., x_n be a random sample of size n from the normal population with mean μ and variance σ^2.We know that the sample mean $\bar{x} = \frac{\sum x_i}{n}$ is sufficient and unbiased estimator for population mean with minimum variance $\frac{\sigma^2}{n}$. The usual practice to compare the estimators based on mean square error (MSE) for location parameter and may not yield a clear favorite for scale parameter. One way to make the problem of finding a ‘best estimator tractable is to limit the class of estimators. A popular way of restricting the class of estimators by consideringunbiased and invariance estimators.

Searls (1964) has suggested the improved estimator $Y' = \frac{n\bar{x}}{n+\vartheta^2}$ in the class of estimators $Y' = c\bar{x}$ and show that

$$MSE\left(Y'\right)=\frac{\sigma^2}{n}\left(1+\frac{\vartheta^2}{n}\right)^{-1} < MSE\left(\bar{x}\right)=\frac{\sigma^2}{n}. \tag{1.1}$$

In negative exponential distribution (N.E.D.) with E(x)=θ, V(x) = θ^2 andϑ=1. The improved estimator is $Y_1=\frac{n\bar{x}}{n+1}$ with $MSE(Y_1)=\frac{\theta^2}{n+1}$ which is smaller then $\frac{\theta^2}{n}$,θ is the scale parameter. In normal distribution having mean μ and variance σ^2, where σ^2 behaves as scale parameter and the maximum likelihood estimate is $S^2=\frac{1}{n}\sum\left(x_1-\bar{x}\right)^2\left(MLE\right)$ and $s^2=\frac{1}{n-1}\sum\left(x_1-\bar{x}\right)^2$ (the unbiased estimator) are the estimators for σ^2.

Thus $MSE\left(S^2\right)=\frac{2\sigma^4}{n}$ and $\mathrm{MSE}\left(s^2\right)=\frac{2\sigma^4}{n-1}$.

Varian (1975) proposed the Linex (linear-exponential) lossfunction. The equation of Linex loss is

$$L\left(\Delta,a\right)=b\left[e^{a\Delta}-a\Delta-1\right],\quad \Delta=\hat{\mu}-\mu,\ a\neq 0, \tag{1.2}$$

Where a and b are shape and scale parameter respectively.
If $|a|\rightarrow 0$, the Linex loss reduce to squared error.

The Linex loss function which rises exponentially on one side of zero and almost linearly on the other side of zero. This loss function reduce to squared error loss for value of a near to zero. Sadooghi(1990) considered theLinex loss for estimating the binomial parameter. Zellner (1986) used this loss function for estimating the mean of a normal distribution. Basu and Ebrahim (1991) considered this loss function in the context of reliability estimation in exponential distribution. Pandey and Rai (1992) considered Bayesian estimation of mean and square of mean of normal distribution using Linex loss function. The sufficient statistics can be used to find the uniformly minimum risk unbiased (UMRU) estimator underLinex loss function (Bell, 1968)). If over-estimation and under-estimation are present in practical situations (just as life testing, quality control, engineering statistics), the Linex loss function can be applied (Pandey, 1997), (Pandey &Srivastava, 2001), (Rojo, 1987), (Zellner, 1986), (Pandey and Rai ,1992).The MMSE criterion is inadmissible under Linex loss function.

In section 2, suggested an improve estimator for mean using Linex loss function and shows that the improved estimator dominates the Searls (1964) estimator underLinex loss function. The sufficient statistics can be used to find the uniformly minimum risk unbiased estimator.
In section 3, an improve estimation forμ^2 is suggested (which uses coefficient of variation) under Linex loss function.
The mathematical expression of improve estimatorof fourth power of mean is also considered in section 4.

In section 5, an improve estimator for common mean in negative exponential distribution is proposed under Linex loss function.

In section 6,Pandey and Malik (1994) considered the $T_1' = w_1\bar{x}^2 + w_2\bar{y}^2 + w_3\bar{x}\,\bar{y}$ for common mean with the restriction $w_1 + w_2 + w_3 = 1$.Here considered the above estimator for $w_1 + w_2 + w_3 \neq 1$. and studied its property

In section 7, considered the displaced exponential distribution under Linex loss function and suggested an improve estimator.

2. Estimation of mean using Linex loss function

Zellner (1968) proposed the Linex loss function

$L(a, \Delta) = b(e^{a\Delta} - c\Delta - 1), \quad \Delta = \hat{\mu} - \mu, \quad a \neq 0$ andif bc=a ,then this function will be equal to $b(e^{a\Delta} - a\Delta - 1)$

The Linex loss function reduce to squared error if $|a| \to 0$.

Basu and Ebrahimi (1991) considered the invariant form of Linex loss for estimating μ.
The invariant form of Linex loss is

$$L(a, \Delta^*) = b(e^{a\Delta^*} - c\Delta^* - 1), \quad \Delta^* = \frac{\hat{\mu}}{\mu} - 1, \quad a \neq 0.$$

$$R(a, \Delta^*) = E[L(a, \Delta^*)] = \frac{a^2}{2}\left[E\left(\frac{c\bar{x}}{\mu} - 1\right)^2 + \frac{a}{3}E\left(\frac{c\bar{x}}{\mu} - 1\right)^3 + \ldots\right]., \text{ where } \hat{\mu} = c\bar{x}$$

$$\frac{2}{a^2}R(a, \Delta^*) = E\left[\left(\frac{\bar{x}}{\mu} - 1\right)^2\right] + \frac{a}{3}E\left[\left(\frac{\bar{x}}{\mu} - 1\right)^3\right] + \ldots. \tag{2.1}$$

Let us consider an estimator $Y_1 = c\bar{x}$ in case of normal distribution with mean μ and variance σ^2.
The invariant form of Linex loss is

$$L(a, \Delta^*) = e^{a\left(\frac{c\bar{x}}{\mu} - 1\right)} - a\left(\frac{c\bar{x}}{\mu} - 1\right) - 1.$$

$$R(a, \Delta^*) = e^{\frac{a^2c^2\vartheta^2}{2n}}\, e^{-a(1-c)} - ac + a - 1.$$

$$\frac{2}{a^2}R(a, \Delta^*) = \frac{ac^3}{3}\left(1 + \frac{3v^2}{n}\right)e^{-a} + \frac{c^2}{2}\left(1 + \frac{v^2}{n}\right)e^{-a} - (2 - a + \frac{a^2}{3} - \frac{a^3}{12}) + 1 - \frac{a}{3} + \frac{a^2}{12} \tag{2.2}$$

In negative exponential distribution, we have,

$$E\left[e^{\frac{a c \bar{x}}{\theta}}\right]=\left(1-\frac{ac}{n}\right)^{-n} \tag{2.3}$$

And

$$R\left(a, \Delta^{*}\right)=\frac{\bar{e}^{a}}{\left(1-\frac{ac}{n}\right)^{n}}-ac+a-1. \tag{2.4}$$

From equation (2.4), we get the minimum value of c as

$$c_{\min}=\frac{n}{a}\left(1-\bar{e}^{\frac{a}{n+1}}\right) \text{(Pandey (1997).} \tag{2.5}$$

The proposed estimator is $Y_1=\frac{n}{a}\left(1-\bar{e}^{\frac{a}{n+1}}\right)\bar{x}$ with $\text{Min R}\left(\text{a}, \Delta^{*}\right)=a-(n+1)\left(a-\bar{e}^{\frac{a}{n+1}}\right)$.

Thus minimum mean squared error is inadmissible under Linex loss function.
Differentiating equation (2.2) with respect to c and equating to zero, we get,
It will be minimum if

$$c_{\min}=\left(\frac{(a-1)(1+\frac{v^2}{n})+\sqrt{(1-a)^2\left(1+\frac{v^2}{n}\right)^2+4(a-\frac{a^2}{2})(1+\frac{3v^2}{n})}}{a(1+\frac{3v^2}{n})}\right) \tag{2.6}$$

For given values of n, $\vartheta \geq 1$ and $0 \leq a \leq 0.6$, the values of c can be obtained. Putting the $c_{\min}$ in equation (2.2) we obtained the minimum risk. Figure 2.1 to 2.3, represent the relative efficiency of the estimator $Y_1$with respect to Y' for ϑ = 1.00(.25)1.50, and n = 5(5)20 and a = .4(.2).8.
The figure shows that if ν ≥1, the estimator perform better for smaller values of n and the values of aupto 2.00.

Pandey and Rai(1992) considered the Bayes estimator for mean and square of population mean of normal distribution under Linex loss function. Pandey (1997) obtained the result for scale parameter in case of negative exponential distribution using invariant version of Linex loss function as $Y_1 = \frac{n}{a}\left(1 - e^{\frac{a}{n+1}}\right)\bar{x}$.

$$Y_1 = \frac{n\bar{x}}{n+1} - \frac{an\bar{x}}{2(n+1)^2} + \frac{an\bar{x}}{6(n+1)^3} ----.$$

We know that $\frac{2n\bar{x}}{\theta}$ follows a chi-square distribution with 2n degrees of freedom (Gamma (1, n)). Bell(1968) defined a modified Bessel function as

$$H_n(2na\bar{x}) = 1 + \frac{2na\bar{x}}{1!n} + \frac{4a^2n^2\bar{x}^2}{2!n(n+1)} + \frac{8n^3a^3\bar{x}^3}{1!n(n+1)(n+2)} + \ldots$$

$$= 1 + 2a\bar{x} + \frac{4na^2\bar{x}^2}{2(n+1)} + \ldots$$

$$E\left[H_n(2na\bar{x})\right] = 1 + 2a\theta + \frac{a^2}{2!}4\theta^2 + \ldots \qquad = e^{2a\theta}.$$

$$\log E[H_n(2na\bar{x})] = 2a\theta \Rightarrow \frac{1}{2a}\log E[H_n(2an\,\bar{x})] = \theta.$$

This MVRU estimator for θ is

$$\hat{\theta} = \bar{x} - \frac{a\,\bar{x}^2}{(n+1)} + \ldots\ldots\ldots\ldots \quad .$$

This shows that sufficient statistics $\bar{x}$ can be used to find UMRU estimator in Linex loss function.

3. Estimation of square of mean usingLinex loss function

In normal distribution, we know that $V(\bar{x}) = \frac{\sigma^2}{n}$ which implies $\hat{\mu}^2 = \frac{\bar{x}^2}{1+\frac{\vartheta^2}{n}}$.

If we consider $Y_2 = t_2\bar{x}^2$, the minimum value of t_2 is

$$t_{2\min} = \frac{1+\frac{\vartheta^2}{n}}{\left(1+\frac{\vartheta^2}{n}\right)^2 + \frac{\vartheta^2}{n}\left(4+\frac{2}{n}\vartheta^2\right)} \le 1$$

Therefore the proposed estimator is $Y_2 = \dfrac{\bar{x}^2}{1+\dfrac{\vartheta^2}{n}\left\{1+\dfrac{4+\dfrac{2\vartheta^2}{n}}{1+\dfrac{\vartheta^2}{n}}\right\}}$ if ϑ is known.

If ϑ is unknown, the MVUE for μ^2 is $U = \bar{x}^2 - \dfrac{s^2}{n}$.

For smaller value of n, U may be negative and Das (1975) suggested a biased estimator for μ^2 as D= $\left(1+\dfrac{s^2}{n\bar{y}^2}\right)^{-1} \bar{x}^2$ and studied its large sample properties. To obtain an estimator which has same mean square error as D for large sample size n but has smaller bias in D, Pandey (1980) suggested an estimator

$$P = \frac{\bar{x}^2}{1+\dfrac{s^2}{n\bar{y}^2}\left(1+\dfrac{s^2}{n\bar{y}^2}\right)}$$

The invariant form of Linex loss function for the estimator $Y_4 = t_4\bar{x}^2$ is

$$L(a,\Delta^*) = e^{-a}e^{\frac{a\bar{x}^2 t_4}{\mu^2}} - a\left(\frac{a\bar{x}^2 t_4}{\mu^2} - 1\right) - 1. \tag{3.1}$$

$$R(a,\Delta^*) = e^{-a}E\left[e^{\frac{a\bar{x}^2 t_4}{\mu^2}}\right] - aE\left(\frac{a\bar{x}^2 t_4}{\mu^2} - 1\right) - 1. \tag{3.2}$$

$$\frac{2}{a^2}R(a,\Delta^*) = \frac{a{t_4}^3}{3}E\left(\frac{\bar{x}^6}{\mu^6}\right)e^{-a} + \frac{{t_4}^2}{2}E\left(\frac{\bar{x}^4}{\mu^4}\right)e^{-a} - (2-a+\frac{a^2}{3}-\frac{a^3}{12})(1+\frac{v^2}{n})t_4 + 1 - \frac{a}{3} + \frac{a^2}{12}.$$

Differentiating this equation w r to t_4 and equating to zero, we have

$$t_{4m} = \frac{-E\left(\dfrac{\bar{x}^4}{\mu^4}\right) + \sqrt{E^2(\dfrac{\bar{x}^4}{\mu^4}) + e^a(2-a+\dfrac{a^2}{3}-\dfrac{a^3}{12})(1+\dfrac{v^2}{n})E\left(\dfrac{\bar{x}^4}{\mu^4}\right)}}{2aE\left(\dfrac{\bar{x}^6}{\mu^6}\right)}.$$

which indicate that $0 \leq t_{4m} \leq 1$.

Pandey and Singh (1977) proposed the improved estimator $Y_5 = c_5 \bar{x}^2, 0 \leq c_5 \leq 1$ in case of negative exponential distribution. In case of N.E.D. with E (θ,θ) we have $E(\bar{x}^2) = \frac{n+1}{n}\theta^2$ and $E(\bar{x}^4) = \frac{(n+3)(n+2)(n+1)}{n^3}\theta^4$.

The invariant form of Linex loss function is

$$R(a,\Delta^*) = e^{\frac{ac_5\bar{x}^2}{\theta^2}} e^{-a} - a\left(\frac{c_5\bar{x}^2}{\theta^2} - 1\right) - 1.$$

which has

$$\frac{2}{a^2}R(a,\Delta^*) = \frac{ac_5^3(n+5)(n+4)(n+3)(n+2)(n+1)}{3n^5} + \frac{(1-a)c_5^2(n+3)(n+2)(n+1)}{n^3} + \frac{2c_5(n+1)(\frac{a}{2}-1)}{n} + 1 - \frac{a}{3}. \quad (3.3)$$

Differentiating this equation with respect to c_5 and equating to zero, we get

$$\frac{ac_5^2(n+5)(n+4)(n+3)(n+2)(n+1)}{n^5} + \frac{2(1-a)c_5(n+3)(n+2)(n+1)}{n^3} + \frac{2(n+1)(\frac{a}{2}-1)}{n} = 0.$$

Again differentiating equation (3.3) with respect to c_5 we get,

$c_5 \geq \frac{(a-1)n^2}{a(n+5)(n+4)}$ and c_5 must lies between $\frac{(a-1)n^2}{a(n+5)(n+4)} \leq c_5 \leq 1$

Differentiatingequation (3.3) with respect to c_5 and equating to zero, we have

$$c_{5\min} = \frac{\frac{2(a-1)(n+3)(n+2)(n+1)}{n^3} + \sqrt{\frac{4(1-a)^2(n+3)^2(n+2)^2(n+1)^2}{n^6} - \frac{8(n+5)(n+4)(n+3)(n+2)(n+1)^2(\frac{a}{2}-1)}{n^6}}}{\frac{2(n+5)(n+4)(n+3)(n+2)(n+1)}{n^5}}.$$

4. Estimation of fourth power of mean under Linex loss function

Let us consider anestimator for the fourth power of mean as $Y_6 = t_6 \bar{x}^4$. We have $E\left(\bar{x}^4\right) = \mu^4 + \frac{6\mu^2\sigma^2}{n} + \frac{3\sigma^4}{n^2} = \mu^4\left[1 + 6\frac{\vartheta^2}{n} + 3\frac{\vartheta^4}{n^2}\right]$ and

$$MSE(Y_6) = t_6 V(\bar{x}^4) + \left[t_6\left\{1 + \frac{6v^2}{n} + \frac{3v^4}{n^2}\right\} - 1\right]^2 \mu^8 .$$

The values of t_6 for which $MSE(Y_6)$ will be minimum can be obtained.

In negative exponential distribution with $v = 1$ and $\frac{2n\bar{x}}{\theta}$ follows the Chi- square with 2n defend

$$E\left(\bar{x}^4\right) = \frac{(n+3)(n+2)(n+1)}{n^3}\theta^4 ,$$

$$\mathrm{V}\left(\bar{\mathrm{x}}^4\right) = \left[\frac{(\mathrm{n}+7)(\mathrm{n}+6)(\mathrm{n}+5)(\mathrm{n}+4)(\mathrm{n}+3)(\mathrm{n}+2)(\mathrm{n}+1)\mathrm{n}}{\mathrm{n}^8} - \frac{(\mathrm{n}+3)^2(\mathrm{n}+2)^2(\mathrm{n}+1)^2\mathrm{n}^2}{\mathrm{n}^8}\right]\theta^8 .$$

$$MSE(Y_6) = t_6^2 V\left(\bar{x}^4\right) + Bias^2\left(t_6\bar{x}^4\right).$$

The minimum value of t_6 for which MSE (Y_6) will be minimum is

$$t_{6\,\min} = \frac{n^4}{(n+7)(n+6)(n+5)(n+4)}. \tag{4.1}$$

The invariant form of Linex loss function in negative exponential distribution is

$$L\left(a, \Delta^*\right) = \bar{e}^a\, e^{\frac{at_6\bar{x}^4}{\theta^4}} - at_6\frac{\bar{x}^4}{\theta^4} + a - 1. \tag{4.2}$$

$$\frac{2}{a^2}R\left(a,\Delta^*\right) = \left(1 - \frac{a}{3} + \frac{a^2}{4.3}...\right) - \left(2 - \frac{a}{1!} + \frac{a^2}{3}...\right)\left(\frac{(n+3)(n+2)(n+1)}{n^3}\right)t_6 + \bar{e}^a$$

$$\frac{(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)}{n^7}t_6^2 + \frac{a}{3}\bar{e}^a$$

$$\frac{(n+11)(n+10)(n+9)(n+8)(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)}{n^{11}}t_6^3 . \tag{4.3}$$

Differentiating this equation with respect to t_6 and equating to zero, we get the value of $t_{6\ min}$.
If $|a| \to 0$, we get the values according to equation (4.1).

5. Estimation of combine mean under Linex loss function

Let $x_i, i = 1, 2, ... n_1$ and y_j $j = 1, 2, ... n_2$ be the random samples of sizes n_1 and $n_2$from two exponential distributions with parameters θ_1 and θ_2 respectively. The combineestimator for mean is

$$Y_2^{'} = \frac{1}{a}\left(1 - e^{\frac{-a}{n_1+n_2+1}}\right)\left(n_1\bar{x} + n_2\bar{y}\right). \tag{5.1}$$

With $Min\, R\left(Y_2\right) = a - \left(n_1 + n_2 + 1\right)\left(1 - \bar{e}^{\frac{a}{(n_1+n_2+1}}\right)$ for pooled estimator under Linex loss function when $\theta_1 = \theta_2 = \theta$.

For squared error $|a| \to 0$ and MMSE estimator is $\hat{\theta}_m = \frac{n_1\bar{x} + n_2\bar{y}}{n_1 + n_2 + 1}$ which is inadmissible under L (Δ^*) (Rai (1996)).

If means of two populations are same but variances are unequal, the estimator for common mean is

$$Y_7 = t_6 \frac{\frac{n_1\bar{x}}{\sigma_1^2} + \frac{n_2\bar{y}}{\sigma_2^2}}{\left(\frac{n_1}{\sigma_1^2} + \frac{n_2}{\sigma_2^2}\right)} \approx t_6\left(\frac{\frac{n_1\bar{x}}{\vartheta_1^2} + \frac{n_2\bar{y}}{\vartheta_2^2}}{\frac{n_1}{\vartheta_1^2} + \frac{n_2}{\vartheta_2^2}}\right) = t_6\left(l_1\bar{x} + l_2\bar{y}\right)., \tag{5.2}$$

where $l_1 = \left(n_1 / \vartheta_1^2\right) / \left(\frac{n_1}{\vartheta_1^2} + \frac{n_2}{\vartheta_2^2}\right)$, $l_2 = 1 - l_1$, ϑ_1 & ϑ_2 are known.

The invariant form of Linex loss function is

$$L\left(a, \Delta^*\right) = \bar{e}^{a}\ e^{at_6}\left(\frac{l_1\bar{x} + l_2\bar{y}}{\mu}\right) - a\left\{t_6\frac{l_1\bar{x} + l_2\bar{y}}{\mu}\right\} + a - 1 \tag{5.3}$$

$$R\left(a, \Delta^*\right) = \bar{e}^{a}\ E\left[e\frac{at_6\ l_1\bar{x}}{\mu}\right].E\left[e\frac{at_6\ l_2\bar{y}}{\mu}\right] - a\ E\left[t_6\ \frac{l_1\bar{x} + l_2\bar{y}}{\mu}\right] + a - 1 \tag{5.4}$$

In normal distribution, we have

$$E\left[e\frac{at_6 l_1 \bar{x}}{\mu}\right] = e^{at_6 l_1} + \frac{a^2 t_6^2\, l_1^2\, \vartheta_1^2}{2n_1},\ E\left[e\frac{at_6 l_2 \bar{y}}{\mu}\right] = e^{at_6 l_2 + \frac{1}{2}}\frac{a^2 t_6^2\, l_2^2\, \vartheta_2^2}{n_2}$$

Therefore from equation (5.4) we have,

$$\frac{2}{a^2}R(a,\Delta^*) = \left(1 - \frac{a}{3} + \frac{a^2}{4.3}..\right) - \left(2 - a + \frac{a^2}{3} - \frac{a^3}{4.3} + ..\right)t_6 + \bar{e}^a(1+p_1)t_6^2 + \left(\frac{a}{3} + ap_1\right)\bar{e}^a t_6^3$$

where

$$p_1 = \frac{1}{\frac{n_1}{\vartheta_1^2} + \frac{n_2}{\vartheta_2^2}}.$$

Differentiating with respect to t_6 and equating to zero, we get

$$-\left(2 - a + \frac{a^2}{3} - \frac{a^3}{4.3}..\right) + 2\bar{e}^a(1+p_1)t_6 + (a + 3p_1 a)\bar{e}^a\, t_6^2 = 0$$

If a = o we have

$$t_6 = \frac{n_1\vartheta_2^2 + n_2\vartheta_1^2}{n_1\vartheta_2^2 + n_2\vartheta_1^2 + \vartheta_1^2\,\vartheta_2^2} \text{ (Pandey\& Singh (1978)).} \tag{5.5}$$

In case of N.E.D. and if, $\vartheta_1 = 1,\ \vartheta_2 = 1$ then $l_1 = \frac{n_1}{n_1 + n_2}$, $l_2 = \frac{n_2}{n_1 + n_2}$ and improved estimator is

$$Y_6 = \frac{n_1\bar{x} + n_2\bar{y}}{n_1 + n_2 + 1}.$$

The improved estimator under Linex loss can be obtained.

6. Estimation of square of common mean in negative exponential distribution

Suppose the estimator for square of common mean μ as

$$Y_7 = \left(\frac{n_1\bar{x} + n_2\bar{y}}{n_1 + n_2}\right)^2 = \frac{n_1^2\bar{x}^2 + n_2^2\bar{y}^2 + 2n_1n_2\bar{x}\,\bar{y}}{(n_1 + n_2)^2}.$$

Pandey and Malik (1994) proposed the estimator for μ^2 is negative exponential distribution under the squared error loss function as

$$T_1' = w_1\bar{x}^2 + w_2\bar{y}^2 + w_3\bar{x}\,\bar{y} \tag{6.1}$$

where w_1, w_2 and w_3 are weights and

$$w_1 = \frac{n_1^2}{(n_1 + n_2)^2 + 5(n_1 + n_2) + 6},\ w_2 = \frac{n_2^2}{(n_1 + n_2)^2 + 5(n_1 + n_2) + 6}$$

$$w_3 = \frac{2n_1n_2}{(n_1 + n_2)^2 + 5(n_1 + n_2) + 6}$$

The improved estimator is

$$T''_1 = \frac{n_1^2\bar{x}^2 + n_2^2\bar{y}^2 + 2n_1n_2\,\bar{x}\,\bar{y}}{(n_1 + n_2)^2 + 5(n_1 + n_2) + 6} \text{ with}$$

$$\text{MSE}(T''_1) = \frac{2(2n_1 + 2n_2 + 3)}{(n_1 + n_2)^2 + 5(n_1 + n_2) + 6}. \tag{6.2}$$

The invariant form of Linex loss for the estimator $Y_8 = t_8\{w_1\bar{x}^2 + w_2\bar{y}^2 + w_3\overline{xy}\}$ is

$$L(a, \Delta^*) = \bar{e}^{a}\, e^{t_8\left\{\frac{w_1\bar{x}^2 + w_2\bar{y}^2 + w_2\overline{xy}}{\mu^2}\right\}} - at_8\left\{\frac{w_1\bar{x}^2 + w_2\bar{y}^2 + w_2\overline{xy}}{\mu^2}\right\} + a - 1. \tag{6.3}$$

If $\overline{x}$ and $\overline{y}$ are independent negative exponential distributions, we have

$$\frac{2}{a^2}R\left(a,\Delta^*\right)=\left(1-\frac{a}{3}+\frac{a^2}{4.3}..\right)-\left(2-a+\frac{a^2}{3}-\frac{a^3}{4.3}+..\right)\left(\frac{n_1(n_1+1)+n_2(n_2+1)+2n_1n_2}{(n_1+n_2)^2+5(n_1+n_2)+6}\right)t_8$$

$$+\frac{\overline{e}^a}{\left[(n_1+n_2)^2+5(n_1+n_2)+6\right]^2}\left[n_1(n_1+1)(n_1+2)(n_1+3)+n_2(n_2+1)(n_2+2)(n_2+3)\right.$$

$$\left.+6n_1n_2(n_1+1)(n_2+1)+4n_1n_2\{n_1(n_1+1)+n_2(n_2+1)\}\right]t_8^2$$

$$+\frac{1}{3}\frac{a\overline{e}^a}{\left[(n_1+n_2)^2+5(n_1+n_2)+6\right]^3}\left[n_1(n_1+1)(n_1+2)(n_1+3)(n_1+4)(n_1+5)+n_2(n_2+1)\right.$$

$$(n_2+2)(n_2+3)(n_2+4)(n_2+5)+8n_1n_2(n_1+1)(n_1+2)(n_2+1)(n_2+2)+12n_1^2n_2^2$$

$$(n_1+1)(n_2+1)+12n_1(n_1+1)n_2^2(n_2+1)^2+6n_1^2n_2(n_1+1)(n_1+2)(n_1+3)+12n_2(n_2+1)$$

$$n_1^2(n_1+1)^2+6n_1n_2^2(n_2+1)(n_2+2)(n_2+3)+3n_1n_2(n_1+1)(n_1+2)(n_1+3)(n_2+1)$$

$$\left.+3n_1n_2(n_1+1)(n_2+1)(n_2+2)(n_2+3)\right]\quad. \tag{6.4}$$

If a=0 (squared error), we have

$$\frac{2}{a^2}R\left(a,\Delta^*\right)=1-2\frac{n_1(n_1+1)+n_2(n_2+1)+2n_1n_2}{(n_1+n_2)^2+5(n_1+n_2)+6}t_8+$$

$$\frac{\left[\begin{array}{r}n_1(n_1+1)(n_1+2)(n_1+3)+n_2(n_2+1)(n_2+2)(n_2+3)+6n_1n_2(n_1+1)(n_2+1)\\4n_1n_2\{n_1(n_1+1)+n_2(n_2+1)\}\end{array}\right]}{\left[(n_1+n_2)^2+5(n_1+n_2)+6\right]^2}t_8$$

Differentiating equation (6.4) with respect to t_8 and equating to zero we get,

$$t_{8\min}=\frac{\left[(n_1+n_2)^2+5(n_1+n_2)+6\right](n_1+n_2)(n_1+n_2+1)}{\left[\begin{array}{l}n_1(n_1+1)(n_1+2)(n_1+3)+n_2(n_2+1)(n_2+2)(n_2+3)+6n_1n_2(n_1+1)(n_2+1)\\+4n_1n_2\{n_1(n_1+1)+n_2(n_2+1)\}\end{array}\right]}$$

If n_2=0, we get $t_{8\min}$ = 1 and the improved estimator is $Y_2=\frac{n_1^2\bar{x}^2}{(n_1+2)(n_1+3)}$

The relative efficiency for different values of a, n_1=5(5) 20 and n_2=5(5)10 20
were calculated in Table 6.1.

The table 6.1, showsthat if a>1 the relative efficiency increases if n_2 increases. Again for fixed n_2 the relative efficiency increases for increasing n_1 of scale and its function.

7.Estimation of displaced exponential distribution underLinex loss function

Let x_1, x_2, ...,x_n be a random sample of size n from a displaced exponential distribution having p.d.f.

$$f(x,A,\theta)=\frac{1}{\theta}\bar{e}\frac{(x-A)}{\theta},\ \ x>A,\ \theta>0 \tag{7.1}$$

Here A is location and θ is scale parameters. The maximum likelihood estimators for θ and A are $(\bar{x}-x_{(1)})$ and $x_{(1)}$ respectively.

We know that $\frac{2n(\bar{x}-x_{(1)})}{\theta}$ follows a chi-square distribution with 2(n-1) degrees of freedom.

Thus $E\{(\bar{x}-x_{(1)})\}=\left(\frac{n-1}{n}\right)\theta, V\{(\bar{x}-x_{(1)})\}=\left(\frac{n-1}{n^2}\right)\theta \Rightarrow \frac{n}{n-1}(\bar{x}-x_{(1)})^2$ is unbiased estimator for θ^2.

The invariant form of Linex function for the estimator $D_1 = l_1^{'}\left(\overline{x} - x_{(1)}\right)$ is

$$L\left(a, \Delta^*\right) = e^{a\frac{\left(l_1^{'}\left(\overline{x}-x_{(1)}\right)\right)}{\theta}} e^{-a} - a\left\{\frac{l_1^{'}\left(\overline{x} - x_{(1)}\right)}{\theta}\right\} + a - 1$$

$$R\left(a, \Delta^*\right) = e^{-a} E\left[e^{a\frac{l_1^{'}\left(\overline{x}-x_{(1)}\right)}{\theta}}\right] - a\left\{l_1^{'}\left(\frac{n-1}{n}\right)\right\} + a - 1$$ .We have

$$E\left[e^{a\frac{l_1^{'}\left(\overline{x}-x_{(1)}\right)}{\theta}}\right] = \left(1 - \frac{a l_1^{'}}{n}\right)^{-(n-1)}$$ and $$R\left(a, \Delta^*\right) = \frac{e^{-a}}{\left(1 - \frac{a l_1^{'}}{n}\right)^{(n-1)}} - a\left\{l_1^{'}\left(\frac{n-1}{n}\right)\right\} + a - 1$$

Differentiating with respect to $l_1^{'}$ and putting equal to zero, we have

$$(n-1)\frac{1}{n} a e^{-a}\left(1 - \frac{a l_1^{'}}{n}\right)^{-(n-1)-1} - a\frac{(n-1)}{n} = 0$$

Thus,

$$e^{-a}\left(1 - \frac{a l_1^{'}}{n}\right)^{-n} = 1 \Rightarrow \overline{e}^{\frac{a}{n}} = 1 - \frac{a l_1^{'}}{n}$$

or, $l_1^{'} = \frac{n}{a}\left(1 - \overline{e}^{\frac{a}{n}}\right)$.

The improved estimator is

$$D_1 = \frac{n}{a}\left(1 - \overline{e}^{\frac{a}{n}}\right)\left(\overline{x} - x_{(1)}\right) = \frac{n}{a}\left[1 - \left(1 - \frac{a}{n} + \frac{a^2}{2n^2} - \right)\right]\left(\overline{x} - x_{(1)}\right)$$

$$= \frac{n}{a}\left[1 - 1 + \frac{a}{n} - \frac{a^2}{2n^2} + ...\right]\left\{\overline{x} - x_{(1)}\right\} = \left(\overline{x} - x_{(1)}\right) - \frac{a}{2n}\left\{\overline{x} - x_{(1)}\right\}^2 + ...$$

Thus $\left(\overline{x} - x_{(1)}\right)$ is improved estimator for θ if $|a| \to 0$ (squared error). We know that

$\frac{n}{n-1}\left\{\overline{x} - x_{(1)}\right\}^2$ is unbiased estimator for θ^2.

8. References

[1] Basu, A.P. and Ebrahimi, N. (1991) "Bayesian approach to life testing and reliabilityestimation using asymmetric loss function."Jour. Stat. Plann. Infer, 29, 21-31.
[2] Bell, W.W. (1968) "Special Function for Scientists and Engineers", London: Van Nostrand.
[3] Das, B.(1975) Estimation of μ^2in normal population,Cal.Stat.Assn.Bull.,24,135-140.
[4] Pandey,B.N.(1980) "On estimation of variance in normal distribution".Jour.Ind.Soc.Agri.Stat.33,1-5.
[5] Pandey, B.N. (1997) "Testimator of the scale parameter of the exponential distribution using
[6] Linexloss function." Comm. Stat.Theo.Meth.,26, 2191-2200.
[7] Pandey, B.N.and Malik, H.J. (1994) "Some improved estimators for common variance of two
[8] populations." Comm. Stat. Theo. Meth., 23(10), 3019-3035.
[9] Pandey, B.N.andRai, Omkar (1992) "Bayesian estimation of mean and squares of mean of
[10] normal population distribution using Linex loss function." Comm. Stat.Theo.Meth.21 (12),
[11] 3369-3391.
[12] Pandey, B.N.and Singh, K.N. (1978) "A pre-test shrinkage estimator of mean of a normalpopulation."Jour.Ind.Soc.Agri.Stat.30, 91-98.
[13] Pandey, B.N.and Singh, J. (1977a) "Estimation of variance of normal population usingapriori information."Jour.Ind. Sta.Assoc.15, 141-150.
[14] Pandey, B.N.and Singh, J. (1977b) "A note on the estimation of variance in exponentialdensity."Sankhya 39, 294-298.
[15] Pandey, B.N.andSrivastava A.K. (2001) "Estimation of variance using asymmetric lossfunction."IAPQR.26 (2), 109-123.
[16] Rai,O.(1996) "A some-times pool estimation of mean life under Linex lossfunction."Comm.Sta.Theo.Meth.25, 2057-2067.
[17] Rojo, J. (1987) "On the admissibility of c x +d with respect to Linex loss function."Comm.
[18] Stat.Theo.Meth.16, 3745-3748.
[19] Sadooghi-Alvandi, S.M. (1990) "Estimation of the parameter of a Poisson distribution using
[20] aLinex loss function."Aust.Jour.Stat.32, 393-398.
[21] Searls, D.T. (1964) "The utilization of a known coefficient of variation in the estimationprocedure."Jour.Amer.Stat.Assoc.59, 1225-1226.
[22] Varian, H.R. (1975) "A Bayesian approach to real estate assessment." In studies in Bayesian
[23] econometrics and statistics in honor of L. J. Savage, Eds S.E. Feinberge and A, Zellner,Amsterdem North Holland, 195-208.
[24] Zellner, A. (1986) "Bayesian estimation and prediction using asymmetric loss function."Jour.Amer.Stat.Assoc.81, 446-451.

Appendices

n \ a	5	10	15	20
.4	16.57	17.74	18.97	20.39
.6	8.6	9.65	11.43	13.57
.8	5.54	6.76	8.54	11.54

Table –2.1Relative efficiency of the estimator Y_1 with respect to Y' for $\vartheta = 1.00$

n a	5	10	15	20
.4	10.31	10.82	11.29	11.78
.6	5.2	5.72	6.24	6.84
.8	3.3	3.74	4.25	4.8

Table –2.2Relative efficiency of the estimator Y_1 with respect to Y' for $\vartheta = 1.25$

a \ n	5	10	15	20
.4	7.03	7.31	7.54	7.77
.6	3.2	3.76	3.99	4.25
.8	2.19	2.41	2.62	2.87

Table –2.3 Relative efficiency of the estimator Y_1 with respect to Y' for $\vartheta = 1.50$

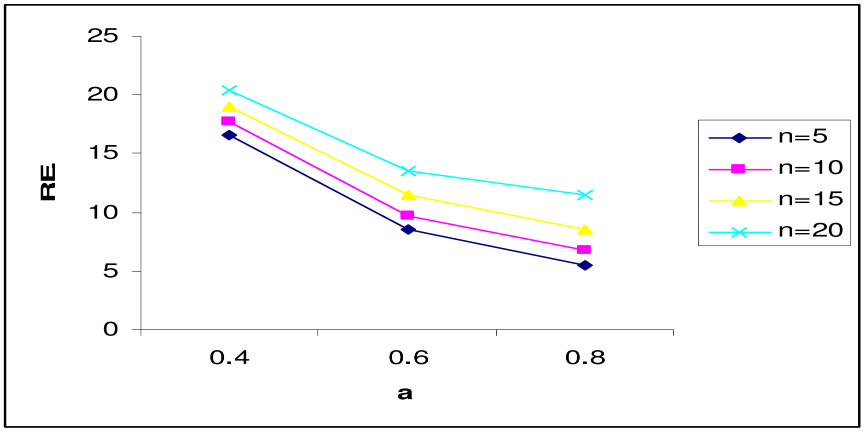

Figure 2.1 Relative efficiency of the estimator Y_1 with respect to Y' for ϑ = 1.00

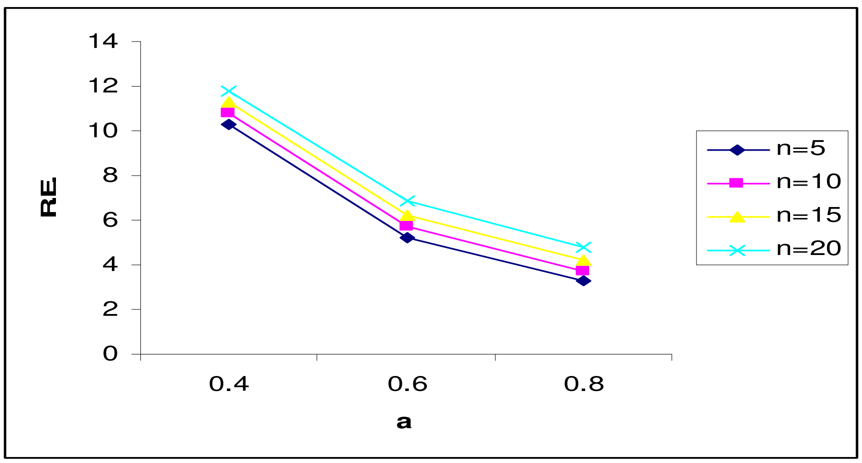

Figure 2.2Relative efficiency of the estimator Y_1 with respect to Y' for ϑ = 1.25

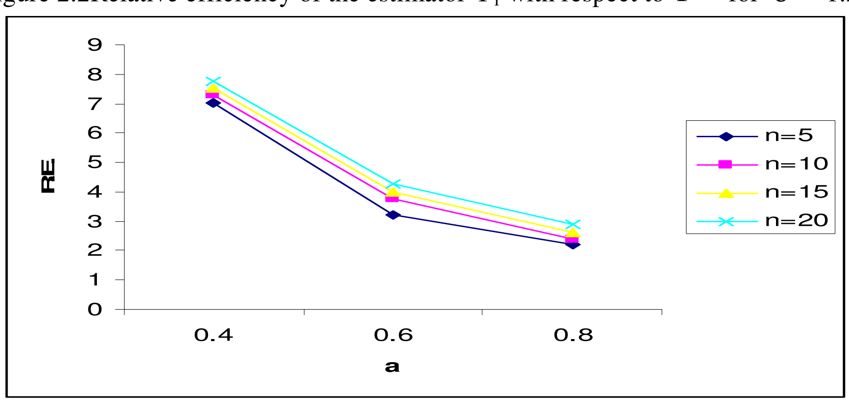

Figure 2.3Relative efficiency of the estimator Y_1 with respect to Y' for ϑ =1.50

n_2 \ n_1		5	10	15	20
5	(1)	1.431	1.638	1.631	1.585
	(2)	1.509	1.744	1.751	1.715
	(3)	1.603	1.877	1.904	1.884
	(4)	1.686	1.990	2.032	2.026
	(5)	1.717	2.016	2.051	2.037
10	(1)	1.638	2.418	2.729	2.789
	(2)	1.744	2.629	3.008	3.108
	(3)	1.877	2.903	3.381	3.544
	(4)	1.990	3.133	3.698	3.918
	(5)	2.016	3.146	3.691	3.890
15	(1)	1.631	2.729	3.478	3.855
	(2)	1.751	3.008	3.916	4.410
	(3)	1.904	3.381	4.528	5.216
	(4)	2.032	3.698	5.061	5.936
	(5)	2.051	3.691	5.001	5.818
20	(1)	1.585	2.789	3.855	4.581
	(2)	1.715	3.108	4.410	5.356
	(3)	1.884	3.544	5.216	6.542
	(4)	2.026	3.918	5.936	7.648
	(5)	2.037	3.890	5.818	7.404

(1) a =.2, (2) a= .4, (3) a = .6, (4) a=.8,(5) a=1.