PRELIMINARY TEST ESTIMATION AND SHRINKAGE PRELIMINARY TEST ESTIMATION IN NORMAL AND NEGATIVE EXPONENTIAL DISTRIBUTION USING LINEX LOSS FUNCTION

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Abstract

In statistical estimation procedure prior information regarding the unknown value of parameter is utilizing and it may result in a decrease of sampling variability of the estimator or it may save sample size which is desirable in many estimation procedures. The commonly used approaches in statistical inference which utilize prior information are Bayesian approach, preliminary test procedure and shrinkage estimation. The paper proposes preliminary test estimator and shrinkage preliminary test estimator for the variance in normal distribution and studies its property under Linex loss function. The paper also proposes and suggests shrinkage preliminary test estimator for the variance in negative exponential distribution and studies its property under Linex loss function.

Key words

Linex loss function, Shrinkage estimation, Preliminary test estimation, Shrinkage preliminary test estimation

1. INTRODUCTION

Thompson (1968) introduces the idea of shrinkage estimation and found that the shrinkage estimator perform better if the guess value is in the vicinity of true value and when sample size is small. In many practical problems it may not be known whether a prior value ($\theta_0$) is close to the true value of the parameter [3]. If $H_0: \theta = \theta_0$ is accepted, then the shrinkage estimator otherwise the usual estimator can be used.[2],[4-8], [10-11], [13], [15-21], [24], [26-28] and [31] have used preliminary test estimator and shrinkage estimator in different distributions. [25] showed that the non-optimality of preliminary test estimator for mean in normal, binomial and Poisson distribution.[14] proposed shrinkage estimator for the mean in an exponential distribution under type II censoring data. [2] extended the above estimator to mean ($\theta$) in an exponential distribution by acceptance region of uniformly most powerful test with a level of significance (for testing the hypothesis [12] used a different weight R which is more conservative than the above in the sense that test statistics is near to the boundary of the critical region if $k = 1$. This suggests that the use of the test statistic for preliminary test estimator in the construction of weight function $k$ is more
reasonable than fixed or pre-determined value of $k$. [9] proposed modified double stage shrinkage estimator.

In the context of real estate assessment, [29] proposed an asymmetric loss function called Linex loss function (linear-exponential) as

$$L(a, \Delta) = e^{a\Delta} - c\Delta - 1 \quad -\infty < \Delta < \infty$$

(1.1)

Where $a, c > 0$.

The Linex loss function is employed in the analysis of several central statistical estimation and prediction problems. The Linex loss function which rises exponentially on one side of zero and almost linearly on the other side of zero. This loss function behaves linearly for large under-estimation errors ($\Delta < 0$), in which case the exponential term vanishes and exponentially for large over-estimation errors ($\Delta > 0$), in which case the exponential term dominates and vanishes when there is no estimation ($\Delta = 0$).

[31] points out that if $c = a$ then equation (1.1) is minimized at $\Delta = 0$. With this restriction equation (1.1) reduces to

$$L(a, \Delta) = e^{a\Delta} - a\Delta - 1 \quad -\infty < \Delta < \infty$$

(1.2)

[31] points out that for negative values of $a$ Linex loss retains its linear-exponential character, though for opposite estimation error, and that for small values of $|a|$ Linex loss is nearly symmetric and approximately proportional to squared error loss. But for larger values of $|a|$ it is quite asymmetric.
An example is given in the field of hydrology with the estimation of peak water level in the construction of the dam. In that case, overestimation represents a conservative error which increases construction costs, while underestimation corresponds to the much more serious error in which overflows might lead to huge damage in the adjacent area.

The Linex loss function (another form) is

\[
L(\alpha, \Delta^*) = b(e^{a\Delta^*} - c\Delta^* - 1), \quad \Delta^* = \frac{\mu}{\mu} - 1, \quad a \neq 0. \tag{1.3}
\]

Where \(a\) and \(b\) are shaped and scale parameter. If Linex loss reduced to square error.

In section 2, proposed preliminary test estimator for variance in normal distribution as

\[
\hat{\sigma}^2_{PT} = \begin{cases} 
\frac{n-1}{2a} s^2 & \text{if } H_0: \sigma^2 = \sigma_0^2 \text{ is accepted} \\
\frac{n-1}{2a} (1-e^{-\frac{n+1}{a}}) s^2 = P & \text{otherwise}
\end{cases}
\]

And studies its properties under Linex loss function.

Here \(\frac{n-1}{2a} s^2\) is the improve estimator in the class of estimator \(Y = cs^2\) under the Linex loss function [23].

In section 3, proposed shrinkage preliminary test estimator for \(\sigma^2\) in normal distribution and studied its property under Linex loss function and also suggests another shrinkage preliminary test estimator for \(\sigma^2\) in normal distribution.

The proposed shrinkage preliminary test estimator for \(\sigma^2\) in normal distribution as

\[
\hat{\sigma}^2_{SP} = \begin{cases} 
\frac{(n-1)s^2}{(\ell_2 - \ell_1)\sigma_0^2} + \left(1 - \frac{(n-1)s^2}{(\ell_2 - \ell_1)\sigma_0^2}\right)\sigma_0^2 & \text{if } \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \\
\frac{n-1}{2a} (1-e^{-\frac{n+1}{a}}) s^2 & \text{otherwise}
\end{cases}
\]

The other proposes shrinkage preliminary test estimator by taking \(k^2 = k_1\) may be defined as
In section 4, proposed shrinkage preliminary test estimator of variance ($\theta^2$) in negative exponential distribution and studied its property under Linex loss function and also suggest another shrinkage preliminary test estimator for the variance($\theta^2$) in negative exponential distribution.

Let us consider a normal distribution with $N(\mu, \sigma^2)$ and also let $\sigma_0^2$ is the prior value of $\sigma^2$, the preliminary test estimator is

$$
\hat{\sigma}_{SPT}^2 = \begin{cases} 
    k_1s^2 + (1-k_1)s_0^2 & \text{if } \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \\
    \frac{n-1}{2a} \left(1 - e^{-\frac{2a}{n+1}s^2}\right) = P & \text{otherwise}
\end{cases}
$$

(1.6)

The other proposes shrinkage preliminary test estimator by taking $k^2 = k_1$ may be defined as

$$
\hat{\theta}_{SPT}^2 = \frac{n^2}{(n+2)(n+3)} \left(1 - \frac{c}{n}\right) \bar{X}^2 = Y, \text{otherwise}
$$

(1.7)

$$
\hat{\theta}_{SPT}^2 = \frac{n^2}{(n+2)(n+3)} \left(1 - \frac{c}{n}\right) \bar{X}^2 = Y, \text{otherwise}
$$

(1.8)

The value of $c$ and, since the magnitude of the shrinkage factor under Linex loss is smaller than the mean square criterion.

2. PRELIMINARY TEST ESTIMATOR FOR VARIANCE IN NORMAL DISTRIBUTION UNDER LINEX LOSS FUNCTION

Let us consider a normal distribution with $N(\mu, \sigma^2)$ and also let $\sigma_0^2$ is the prior value of $\sigma^2$, the preliminary test estimator is

$$
\hat{\sigma}_{PT}^2 = \begin{cases} 
    \frac{\sigma_0^2}{\sigma_0^2} & \text{if } \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \\
    \frac{n-1}{2a} \left(1 - e^{-\frac{2a}{n+1}s^2}\right) = P, \text{otherwise}
\end{cases}
$$

(2.1)
Here, \( P[\chi^2_{n-1} \leq \ell_1] = P[\chi^2_{n-1} \geq \ell_2] = \alpha/2 \).

The improve estimator of \( \sigma^2 \) under the Linex loss function is

\[
\frac{n-1}{2a} - \frac{2a}{n+1}s^2 = \left[ \frac{n-1}{n+1} - \frac{a(n-1)}{(n+1)^2} + \cdots \right] s^2 = \text{P} [23]
\]

\( \hat{\sigma}^2_{\rho_T} \) under mean square error was considered previously. The risk under an invariant form of Linex loss function is defined as

\[
R(a, \Delta^*) = \mathbb{E}[L(a, \Delta^*)] = \mathbb{E} \left[ \frac{a(\sigma_0^2 - \sigma^2)}{\sigma^2} - a \left( \frac{\sigma_0^2}{\sigma^2} - 1 \right) \right] \cdot \Pr \left[ \ell_1 \sigma_0^2 \leq s^2 \leq \frac{\ell_2 \sigma_0^2}{n-1} \right]
\]

\[
+ \mathbb{E} \left[ \frac{a(P^2/\sigma^2)}{\sigma^2} - a \left( \frac{P}{\sigma^2} - 1 \right) \right] \cdot \Pr \left[ s^2 \leq \frac{\ell_1 \sigma_0^2}{n-1} \text{ or } s^2 \geq \frac{\ell_2 \sigma_0^2}{(n-1)} \right]
\]

(2.2)

Putting \( \frac{(n-1)s^2}{2\sigma^2} = t \), \( s^2 = \frac{2\sigma^2}{n-1}t \), \( ds^2 = \left( \frac{2\sigma^2}{n-1} \right) dt \). Thus,

\[
\frac{2}{a^2} R(a, \Delta^*) = \frac{1}{\text{Gamma} \left[ \frac{n-1}{2} \right]} e^{-\frac{t}{2}} \left[ \frac{\ell_2 \sigma_0^2}{2\sigma^2(n-1)} \right] e^{\frac{n-1}{2} dt}
\]

\[
+ \int_0^\infty \frac{\ell_1 \sigma_0^2}{2\sigma^2(n-1)} \left[ \frac{P}{\sigma^2} - 1 \right] \left[ \frac{a}{3} \left( \frac{P}{\sigma^2} - 1 \right)^3 + \frac{a^2}{4.3} \left( \frac{P}{\sigma^2} - 1 \right)^4 + \cdots \right] \frac{1}{\text{Gamma} \left[ \frac{n-1}{2} \right]} e^{-\frac{t}{2}} \left[ \frac{n-1}{2} \right] dt
\]

Let \( \frac{\sigma_0^2}{\sigma^2} = \delta \), \( \frac{P}{\sigma^2} = \left[ \frac{n-1}{n+1} \frac{a(n-1)}{(n+1)^2} + \cdots \right] \frac{2}{(n-1)} t = \left[ \frac{2}{n+1} - \frac{2a}{(n+1)^2} + \cdots \right] t \)

Thus,
\[\frac{2}{a^2} R(a, \Delta^*) = \left(\delta - 1\right)^2 \frac{2a}{3(n+1)^2} + \frac{a^2}{4.3} (\delta - 1)^4 + \cdots \right] \int_0^{\ell_2/2(n-1)} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-t/2} \frac{n-1}{2} dt + \left(\int_0^{\infty} - \int_0^{\ell_2/2(n-1)} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-t/2} \frac{n-1}{2} dt \right) \]

\[\left[\left\{\frac{2a}{(n+1)^2} + \cdots \right\} + \frac{a^2}{4.3} (\delta - 1)^4 + \cdots \right] + \frac{a^2}{4.3} \left\{\frac{n-1}{n+1} + \frac{2a}{(n+1)^2} + \cdots \right\} e^{-t/2} \frac{n-1}{2} dt\]

If \(|a| \to 0\), thus

\[\frac{2}{a^2} R(a, \Delta^*) = \left(\delta - 1\right)^2 \frac{2(n-1)}{\ell_1/\ell_2} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-t/2} \frac{n-1}{2} dt - \left(\int_0^{\infty} - \int_0^{\ell_2/2(n-1)} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-t/2} \frac{n-1}{2} dt \right) \]

\[= \left(\delta - 1\right)^2 \frac{2(n-1)}{\ell_1/\ell_2} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-t/2} \frac{n-1}{2} dt - \left(\int_0^{\infty} - \int_0^{\ell_2/2(n-1)} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-t/2} \frac{n-1}{2} dt \right) \]

(2.3)

If \(\frac{\sigma_0^2}{\sigma^2} = 1 \Rightarrow \delta = 1\) and \(\frac{\ell_2}{\ell_1} = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-t/2} \frac{n-1}{2} dt = 1 \Rightarrow \ell_1 = 0, \ell_2 = \infty\)

Then, \(\frac{2}{a^2} R(a, \Delta^*) = 0\)

The relative efficiency of estimator \(\hat{\sigma}_{PF}^2\) with respect to \(\hat{\sigma}\) calculated for \((\sigma_0^2/\sigma^2) = \delta = 0.6 (, .2) (1.2), \alpha=5\%, \beta= .2 (.2)1.0 \text{ and } n = 5 (5)15 \text{ in table from 2.1 to 2.4. The tables show that the preliminary test estimator } \hat{\sigma}_{PF}^2 \text{ performs better if } 0.2 \leq a \leq 1 \text{ and } n \text{ is less than 20. The maximum result is at the point } \delta = 1.\)

3. SHRINKAGE PRELIMINARY TEST ESTIMATOR FOR VARIATION IN A NORMAL DISTRIBUTION UNDER LINEX LOSS FUNCTION

[27] and[28] introduced shrunken estimator in life testing distribution. [21] introduced preliminary test estimator for the variance in normal distribution as
\[
\hat{\sigma}^2 = \begin{cases} 
ks^2 + (1-k)\sigma_0^2 & \text{if } H_0 : \sigma^2 = \sigma_0^2 \text{ is accepted} \\
\frac{n-1}{n+1} & \text{otherwise}
\end{cases} 
\]  
(3.1)

Let us consider the class of estimator \( Y = Ws^2 \) and find the value of \( W \) for which \( \text{MSE}(Y) \) is minimized, thus \( W = \frac{n-1}{n+1} \) and the improve estimator is \( Y = \frac{(n-1)s^2}{n+1} \) with \( \text{MSE}(Y) = \frac{2\sigma^4}{n+1} \).

[31] considered Linex loss function, which performs better under the class of estimator \( Y' = W's^2 \) where \( W' = \frac{n-1}{2a} \left( 1 - e^{-\frac{2a}{n+1}} \right) \) where \( a \neq 0 \).

Let us consider shrinkagepreliminary test estimator for the variance in normal distribution as

\[
\hat{\sigma}^2_{SPT} = \begin{cases} 
ks^2 + (1-k)\sigma_0^2 & \text{if } \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \\
\frac{n-1}{2a} \left( 1 - e^{-\frac{2a}{n+1}} \right)s^2 = P & \text{otherwise}
\end{cases} 
\]  
(3.2)

Here \( P[\chi^2_{n-1} \leq \ell_1] = P[\chi^2_{n-1} \geq \ell_2] = \alpha/2 \).

(3.3)

Thus \( \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \Rightarrow \theta \leq \frac{(n-1)s^2}{(\ell_2 - \ell_1)\sigma_0^2} = \frac{\ell_1}{(\ell_2 - \ell_1)} \leq 1 \).

Let us suppose that shrunken factor \( k = \frac{(n-1)s^2}{(\ell_2 - \ell_1)\sigma_0^2} - \frac{\ell_1}{(\ell_2 - \ell_1)} \).

(3.4)

And \( k_1 = \left( \frac{(n-1)s^2}{(\ell_2 - \ell_1)\sigma_0^2} - \frac{\ell_1}{(\ell_2 - \ell_1)} \right)^2 \)

[12] suggested that the value of \( k \) may be taken as the function of test statistics.

Since, \( \frac{(n-1)s^2}{\sigma_0^2} \) is the test criterion for testing of hypothesis \( H_0 : \sigma^2 = \sigma_0^2 \) against \( H_1 : \sigma^2 = \sigma_0^2 \).

The acceptance region has received from
\[
\Pr \left[ \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \right] = 1 - \alpha \quad \Rightarrow \quad \Pr \left[ \frac{\ell_1 \sigma_0^2}{n-1} \leq s^2 \leq \frac{\ell_2 \sigma_0^2}{n-1} \right] = 1 - \alpha.
\]

Thus the proposed shrunken preliminary test estimators are

\[
\hat{\sigma}_{SP1}^2 = \begin{cases} 
\left( \frac{(n-1)s^2}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) s^2 + \left( \frac{1}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) \left( \frac{\ell_1 \sigma_0^2}{n-1} \right) & \text{if } \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \\
\frac{n-1}{2a} \left( 1 - e^{\frac{2a}{n+1}} \right) s^2 & \text{otherwise}
\end{cases}
\]

And

\[
\hat{\sigma}_{SP2}^2 = \begin{cases} 
\left( \frac{(n-1)s^2}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) s^2 + \left( \frac{1}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) \left( \frac{\ell_1 \sigma_0^2}{n-1} \right)^2 & \text{if } \ell_1 \leq \frac{(n-1)s^2}{\sigma_0^2} \leq \ell_2 \\
\frac{n-1}{2a} \left( 1 - e^{\frac{2a}{n+1}} \right) s^2 & \text{otherwise}
\end{cases}
\]

\[
\text{Risk}(\hat{\sigma}_{SP1}^2) = \frac{\frac{\ell_1 \sigma_0^2}{n-1}}{\frac{\ell_1 \sigma_0^2}{n-1} - \frac{\ell_1 \sigma_0^2}{n-1} - \frac{\ell_1 \sigma_0^2}{n-1} \frac{\ell_1 \sigma_0^2}{n-1} \frac{\ell_1 \sigma_0^2}{n-1}} - 1
\]

\[
= \exp \left\{ \left( \frac{(n-1)s^2}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) s^2 + \left( \frac{1}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) \left( \frac{\ell_1 \sigma_0^2}{n-1} \right)^2 \right\}
\]

\[
\int_0^{\infty} \exp \left\{ \left( \frac{(n-1)s^2}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) s^2 + \left( \frac{1}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) \left( \frac{\ell_1 \sigma_0^2}{n-1} \right)^2 \right\} - 1
\]

\[
= \exp \left\{ \left( \frac{(n-1)s^2}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) s^2 + \left( \frac{1}{\sigma_0^2} - \frac{\ell_1}{\ell_2 - \ell_1} \right) \left( \frac{\ell_1 \sigma_0^2}{n-1} \right)^2 \right\}
\]

\[
= \frac{1}{n-1} \left( \frac{(n-1)s^2}{\sigma_0^2} \right)^{\frac{n-1}{2}} d\sigma^2
\]

If \(\frac{(n-1)s^2}{2\sigma^2} = t\), then
The AIC information suggest that the point \( \delta = 1 \). The AIC information suggest that \( \alpha \) should be 16%. The results of 16% can be calculated, but the above recommendation will give useful results.

The relative efficiency of estimator \( \hat{\sigma}_{SP1}^2 \) with respect to P is calculated for \( \delta = 0.6 \) (. 2) (1.2), \( \alpha = 5% \), \( \alpha = 2 \) (. 2)1 and \( n = 5 \) (5)15 in the table from 3.1 to 3.4. The tables show that the estimator \( \hat{\sigma}_{SP1}^2 \) performs better if \( 0.6 \leq \delta \leq 1.2, \alpha = 5% \) and for smaller values of \( n \). The maximum result is at the point \( \delta = 1 \). The AIC information suggest that \( \alpha \) should be 16%. The results of 16% can be calculated, but the above recommendation will give useful results.

The relative efficiency of estimator \( \hat{\sigma}_{SP2}^2 \) with respect to P may also calculate for \( \delta = 0.6 \) (. 2) (1.2), \( \alpha = 5% \), \( \alpha = 2 \) (. 2)1.0 and \( n = 5 \) (5)15.

4. SHRINKAGE PRELIMINARY TEST ESTIMATOR FOR VARIANCE IN NEGATIVE EXPONENTIAL DISTRIBUTION

[22] were considered shrinkage estimator for the variance in negative exponential distribution as

\[
T = k\bar{x}^2 + (1-k)\theta_0^2 \quad , 0 \leq k \leq 1
\]

(4.1)

[21] obtained an improved estimator for \( \theta^2 \) in the class of estimators \( \hat{\theta}_1 = c_1\bar{x}^2 \) under mean square criterion as \( \hat{\theta}_1' = \frac{n^2\bar{x}^2}{(n+2)(n+3)} \) with \( \text{MSE}(\hat{\theta}_1') = \frac{(4n+6)\theta^4}{n(n+1)} \).

The invariant form of the Linex loss function in the class of the estimator \( \hat{\theta}_1 = c_1\bar{x}^2 \) is

\[
L(\alpha, \Delta^*) = e^{a(c_1\bar{x}^2 - 1)} - a \left( \frac{c_1\bar{x}^2}{\theta^2} - 1 \right) - 1 \quad (4.2)
\]
Which has the risk \( R(a, \Delta^*) = E[L(a, \Delta^*)] = e^{-a} E\left[e^{\frac{ac_1 x^2}{\theta^2}}\right] - \frac{ac_1 n}{n} + a - 1 \).

\[
R(a, \Delta^*) = e^a E\left[1 + \frac{ac_1 x^2}{\theta^2} + \frac{a^2 c_1^2 x^4}{2! \theta^4} + \cdots\right] - \frac{ac_1 (n+1)}{n} + a - 1
\]

\[
R(a, \Delta^*) = \left(1 - a + \frac{a^2}{2} - \frac{a^3}{3!} + \frac{a^4}{4!}\right)\left[1 + \frac{ac_1 (n+1)}{n} + \frac{a^2 c_1^2 (n+3)(n+2)(n+1)}{2! n^3} + \cdots\right] - \frac{ac_1 (n+1)}{n} + a - 1
\]

If \(|a| \to 0\)

\[
\frac{2}{a^2} R(a, \Delta) = 1 - \frac{2c_1 (n+1)}{n} + \frac{c_1^2 (n+3)(n+2)(n+1)}{n^3}.
\]

Differentiating with respect to \(c_1\), thus

\[
c_1 = \frac{n^2}{(n+2)(n+3)}.
\]

For other values of \(a\), value of \(c_1\) for different values of \(n\) can be obtained by using quadratic equation. Certainty the value will be smaller than the minimum value under mean square criterion. Let us suppose that

\[
c_1^* = \frac{n^2 \overline{x}^2}{(n+2)(n+3)} \left(1 - \frac{c}{n}\right)\text{ Where } c\text{ is a constant and less than } n.
\]

The propose estimator is

\[
\hat{\theta}_SPT^2 = \begin{cases} 
k \overline{x}^2 + (1-k) \theta_0^2 & \text{if } \theta = \theta_0 \text{ is accepted} \\
\frac{n^2 \overline{x}^2}{(n+2)(n+3)} \left(1 - \frac{c}{n}\right) = Y & \text{otherwise}
\end{cases}
\]

Since \(\frac{2n \overline{x}}{\theta^2}\) follows the chi - square distribution with \(2n\) def. The acceptance region can be defined by

\[
P\left[\ell_1 \leq \frac{2n \overline{x}}{\theta_0^2} \leq \ell_2\right] = 1 - \alpha \quad \text{and} \quad P[\chi_{2n}^2 \leq \ell_1] = P[\chi_{2n}^2 \geq \ell_2] = \alpha/2
\]

Where the value of \(k\) is
\[
k = \frac{2n\bar{x} - \ell_1}{\ell_2 - \ell_1},
\]
which shows that \(0 \leq k \leq 1\).

After squaring \(k\) then the shrinkage factor will be smaller than the previous value. Hence
\[
k_1 = \left(\frac{2n\bar{x} - \ell_1}{\ell_0(\ell_2 - \ell_1)}\right)^2
\]
can be taken.

One may also consider the square of previous shrinkage factor \(k\) which will be smaller than the previous shrunken factor because \(k\) lies between 0 and 1.

The proposed shrinkage preliminary test estimators for \(\theta^2\) in a negative exponential distribution as
\[
\hat{\theta}^2_{SP\bar{T}} = \begin{cases} 
\left(\frac{2n\bar{x} - \ell_1}{\ell_2 - \ell_1}\right)^2 + \left(1 - \left(\frac{2n\bar{x} - \ell_1}{\ell_2 - \ell_1}\right)\right)\hat{\theta}^2_0 & \text{if } \frac{\ell_1\theta_0}{2n} \leq \bar{x} \leq \frac{\ell_2\theta_0}{2n} \\
\frac{n^2(1-c/n)\bar{x}^2}{(n+2)(n+3)} & \text{otherwise}
\end{cases}
\]

And
\[
\hat{\theta}^2_{SP'_2} = \begin{cases} 
\left(\frac{2n\bar{x} - \ell_1}{\ell_2 - \ell_1}\right)^2 + \left(1 - \left(\frac{2n\bar{x} - \ell_1}{\ell_2 - \ell_1}\right)\right)^2\hat{\theta}^2_0 & \text{if } \frac{\ell_1\theta_0}{2n} \leq \bar{x} \leq \frac{\ell_2\theta_0}{2n} \\
\frac{n^2(1-c/n)\bar{x}^2}{(n+2)(n+3)} & \text{otherwise}
\end{cases}
\]

The relative efficiency of estimator \(\hat{\theta}^2_{SP\bar{T}}\) with respect to \(Y\) is calculated for \(z = 0.6(2) (1.2), \alpha=5\%, a=0.2, 4, 6, 1, n = 3, 5, 7\) and \(c = 1, 2\) in the table from 4.1 to 4.8. The table shows that the

59
estimator $\hat{\theta}^2_{SP\hat{T}}$ performs better if $0.6 \leq z \leq 1.2$, for smaller values of $a$ and $n$ under Linex loss function. The maximum result is at the point $z = 1$.

The relative efficiency of estimator $\hat{\theta}^2_{SP\hat{T}}$ with respect to $Y$ may also calculate for $z = 0.6$ (2) (1.2), $\alpha = 5\%$, $a = 0.2, 0.4, 0.6, 1$, $n = 3, 5, 7$ and $c = 1, 2$.

5. SCOPE FOR FURTHER RESEARCH

In this paper author has proposed two estimators for further study. Here author proposed shrinkage preliminary test estimator for the variance in normal distribution and shrinkage preliminary test estimators for the variance ($\theta^2$) in negative exponential distribution.

6. REFERENCES

7. APPENDICES

Table 2.1: Relative Efficiency of estimator $\hat{\sigma}^2_{PT}$ w.r.to P when $\delta = 0.6$ and $\alpha = 5\%$

<table>
<thead>
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Table- 2.2: Relative Efficiency of estimator $\hat{\sigma}_{pr}^2$ w.r.to $P$ when $\delta = 0.8$ and $\alpha = 5\%$

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Table-2.3: Relative Efficiency of estimator $\hat{\sigma}_{pr}^2$ w.r.to $P$ when $\delta = 1.0$ and $\alpha = 5\%$

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Table-2.4: Relative Efficiency of estimator $\hat{\sigma}_{pr}^2$ w.r.to $P$ when $\delta = 1.2$ and $\alpha = 5\%$

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Table-3.1: Relative Efficiency of estimator $\hat{\sigma}_{SPT}^2$ w.r.to $P$ when $\delta = 0.6$ & $\alpha = 5\%$

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Table-3.2: Relative Efficiency of estimator $\hat{\theta}_2^{SPT}$ w.r.to $P$ when $\delta = 0.8$ & $\alpha = 5\%$

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Table-3.3: Relative Efficiency of estimator $\hat{\theta}_2^{SPT}$ w.r.to $P$ when $\delta = 1.0$ & $\alpha = 5\%$

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Table-3.4: Relative Efficiency of estimator $\hat{\theta}_2^{SPT}$ w.r.to $P$ when $\delta = 1.2$ & $\alpha = 5\%$

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Table 4.1: Relative Efficiency of Estimator $\hat{\theta}_{SPT}^2$ w.r.to $Y$ when $\alpha = 0.2$, $c = 1$ & $\alpha = 5\%$

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Table 4.2: Relative Efficiency of estimator $\hat{\theta}_{SPT}^2$ w.r.to $Y$ when $\alpha = 0.4$, $c = 1$ & $\alpha = 5\%$

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Table 4.3: Relative Efficiency of estimator $\hat{\theta}_{SPT}^2$ w.r.to $Y$ when $\alpha = 0.6$, $c = 1$ & $\alpha = 5\%$

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Table-4.5: Relative Efficiency of estimator $\hat{\theta}^2_{SP_{I}}$ w.r.t. $Y$ when $a = 0.2$, $c = 2$ & $\alpha = 5\%$

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Table-4.6: Relative Efficiency of estimator $\hat{\theta}^2_{SP_{I}}$ w.r.t. $Y$ when $a = 0.4$, $c = 2$ & $\alpha = 5\%$

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Table-4.7: Relative Efficiency of estimator $\hat{\theta}_{SPT}^2$ w.r.to $Y$ when $a = 0.6, c = 2$ & $\alpha = 5\%$

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Table-4.8: Relative Efficiency of estimator $\hat{\theta}_{SPT}^2$ w.r.to $Y$ when $a = 1.0, c = 2$ & $\alpha = 5\%$

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