EDGE-TENACITY IN CYCLES AND COMPLETE GRAPHS

Abolfazl Javan, Mina Akhavan, Ali Moeini and Dara Moazzami

Department of Algorithms and Computation, Faculty of Engineering Science, School of Engineering, University of Tehran, Tehran, Iran

ABSTRACT

It is well known that the tenacity is a proper measure for studying vulnerability and reliability in graphs. Here, a modified edge-tenacity of a graph is introduced based on the classical definition of tenacity. Properties and bounds for this measure are introduced; meanwhile edge-tenacity is calculated for cycle graphs and also for complete graphs.

KEYWORDS

Graph, Edge Tenacity

1. INTRODUCTION

Since that network components may be destroyed, it is necessary to study networks vulnerability. So far various criteria have been presented to measure the vulnerability and reliability of networks [9]. All measures have been based on that how the loss of part of the graph (vertices or edges) will affect the graph structure in the worst case. For example, vertex-connectivity is the minimum number of vertices whose removal disconnects the graph and edge-connectivity is the smallest number of edges whose removal disconnects the graph. Integrity [7] and Toughness [8] is two other measures for calculating network vulnerability.

For the first time graph tenacity is studied by Cozzens, Moazzami and Stueckle [2, 3]. Cozzens et al. introduced two measures of network vulnerability. One of them is the tenacity (T(G)) that is introduced for studying vertex vulnerability. The other is the measure that is called mix-tenacity by Moazzami and has been used for studying edge vulnerability. In [4] Moazzami studied integrity, connectivity, toughness and tenacity in some graphs. His results showed that tenacity is a better measure for studying vulnerability and reliability in the most graphs.
The vertex tenacity of a graph called $T_v(G)$ is defined as:

Vertex Tenacity: $$T_v(G) = \min_{A \subseteq V(G)} \{\frac{|A| + \tau(G-A)}{\omega(G-A)}\} \quad (1)$$

where A is a subset V(G) the set of all vertices of the graph G, $\tau(G-A)$ is the number of vertices in the largest component of $(G-A)$ and $\omega(G-A)$ is the number of components of $(G-A)$. Also mix-tenacity is defined as:

Mixed Tenacity: $$T_m(G) = \min_{A \subseteq E(G)} \{\frac{|A| + \tau(G-A)}{\omega(G-A)}\} \quad (2)$$

where A is a subset of E(G), the set of all edges of the graph G, τ(G − A) is the number of vertices in the largest component of G-A and ω(G − A) is the number of components of (G − A). In [5, 6] Piazza et al. used (2) as edge-tenacity but this vulnerability measure is the combination of edges subset ($A \subseteq E(G)$) and the number of vertices in the largest component (τ (G − A)). It is visible that in T_m (Mix-tenacity) the number of removed edges is added to the number of vertices in a largest component of the remaining graph. In this definition both edges and vertices are involved which is not rational. To eliminate this weakness, Moazzami and Salehian [1] presented a new measure and named it edge-tenacity that in new one just edges are involved. Edge-tenacity is defined as:

Edge Tenacity: $$T_e(G) = \min_{A \subseteq E(G)} \{\frac{|A| + \tau(G-A)}{\omega(G-A)}\} \quad (3)$$

where A is a subset of E(G), the set of all edges of the graph G, τ(G − A) is the number of edges in the largest component of (G − A) and ω(G − A) is the number of components of (G − A).

2. EDGE-TENACITY OF SOME SPECIAL GRAPHS

Here, we calculate precisely the edge-tenacity for complete graphs and cycle graphs based on (3).

Theorem 1: If C_n be the cycle graph with n vertices, then

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Proof: Suppose that the number of edges and vertices in C_n are |E|=m and |V|=n respectively. We know that the number of edges and vertices in C_n are equal (m=n). Also it can be easily understood that with removing ω edges, (ω >0), ω components remain in the graph, where ω is a constant. Based on (3) to determine the minimum value of T_e, the numerator of the fraction ω + τ(G − ω) must become minimum. Since τ(G − ω) is the number of edges in the largest component then for getting the minimum value of T_e, it is necessary to distribute |E| − ω edges between ω components equally. Therefore T_e is as:

$$T_e(C_n) = \min_{1 \le \omega \le m} \left\{ \frac{\omega + \left\lceil \frac{m-\omega}{\omega} \right\rceil}{\omega} \right\} = \min_{1 \le \omega \le m} \left\{ 1 + \frac{\left\lceil \frac{m-\omega}{\omega} \right\rceil}{\omega} \right\}. \quad (4)$$

Since |E|=m is constant, it needs that ω changes between 1 to m. Based on (4) one easily conclude that in this range, the minimum value of T_e will achieve when ω is maximum, namely n (m=n). Therefore, T_e is calculated as:

$$T_e(C_n) = 1 + \frac{\left\lceil \frac{m-m}{m} \right\rceil}{m} = 1 + \frac{\left\lceil \frac{n-n}{n} \right\rceil}{n} = 1. \tag{5}$$

Theorem 2: If K_n is a complete graph with n vertices, then

$$T_e(K_n) = \frac{n-1}{2}. \tag{6}$$

For calculation of the edge-tenacity in a complete graph, we first prove some lemmas and then edge-tenacity is calculated.

Lemma 1: The edge-tenacity for the complete graph K_n is calculated in the case that the generated components after removing edges are all complete subgarphs.

Proof: The proof is based on contradiction. Suppose that the edge-tenacity for the complete graph is calculated in the case that at least one of the generated components is not a complete subgraph. Therefore, this component has at least one edge less than the complete graph. By adding this edge to the component if the component is the largest component of the graph, the number of removed edges will be one less and the number of edges of the largest component will be one more that finally does not have any effect on obtaining the edge-tenacity. Also, by adding the removed edge to the component if the component is not the largest one in the graph, the number of removed edges is decreased by one. It means the new value of edge-tenacity is less than the assumed value which contradicts with the assumption of the theorem. Therefore, we can conclude that in order to calculate the edge-tenacity of a complete graph, all generated components of it, after removing edges, can be considered complete.

Lemma 2: Assume that it is supposed to distribute n vertices between ω complete components. The i^{th} component has c_i vertices so that

$$0 < c_1 \le c_2 \le \ldots \le c_\omega, \quad n \ge \omega \tag{7}$$

Where c_i is the number of allocated vertices to i^{th} component that $1 \le i \le \omega$. The maximum value of the total number of edges for all complete components which can be stated by the following equation

$$\sum_{i=1}^{\omega} \binom{c_i}{2}, \tag{8}$$

Occurs when the following conditions are satisfied:

$$c_i = 1, \quad 1 \le i \le \omega - 1, \quad c_\omega = n - \omega + 1 \tag{9}$$

(Note that $\binom{c_i}{2}$ is the number of edges of i[th] complete component and therefore (8) is the total number of edges for all complete components)

Proof: The proof is based on contradiction. Suppose that there exists an integer called k which makes the value in expression (8) maximum such that $c_k>1$, $k<\omega$.

It is known that

$$\binom{c_k}{2}+\binom{c_\omega}{2}=\frac{c_k(c_k-1)}{2}+\frac{c_\omega(c_\omega-1)}{2}=\frac{c_k^2-c_k+c_\omega^2-c_\omega}{2}, \tag{10}$$

If $c_k>1$ then due to (7) $c_\omega>1$ and in consequence we have

$$(c_\omega-1)(c_k-1)>0 \tag{11}$$

By inserting the above in (10) we have

$$\begin{aligned}\binom{c_k}{2}+\binom{c_\omega}{2}&=\frac{c_k^2-c_k+c_\omega^2-c_\omega}{2}<\frac{c_k^2-c_k+c_\omega^2-c_\omega}{2}+(c_\omega-1)(c_k-1)\\&=\frac{c_k^2-c_k+c_\omega^2-c_\omega+2c_kc_\omega-2c_k-2c_\omega+2}{2}=\frac{(c_\omega+c_k-1)(c_\omega+c_k-2)}{2}\\&=\binom{c_\omega+c_k-1}{2}.\end{aligned} \tag{12}$$

Rearranging (8) using (12) yields

$$\begin{aligned}\sum_{i=1}^{\omega}\binom{c_i}{2}&=\left(\sum_{i=1}^{k-1}\binom{c_i}{2}\right)+\binom{c_k}{2}+\left(\sum_{j=k+1}^{\omega-1}\binom{c_j}{2}\right)+\binom{c_\omega}{2}\\&<\left(\sum_{i=1}^{k-1}\binom{c_i}{2}\right)+\left(\sum_{j=k+1}^{\omega-1}\binom{c_j}{2}\right)+\binom{c_\omega+c_\omega-1}{2}+\binom{1}{2}.\end{aligned} \tag{13}$$

The right side of (13) is the total number of edges for all complete components in the case that pick (c_k-1) vertices from k[th] component and place them to ω[th] component. Therefore, the number of edges of k[th] component will be $\binom{1}{2}$ and the number of edges of ω[th] component will be $\binom{c_\omega+c_\omega-1}{2}$. In other word, picking (c_k-1) vertices from k[th] component and placing them to ω[th] component makes (8) greater. So (13) contradicts with the assumption that (8) reaches to its maximum value with the above k. Finally, it can be concluded that the maximum value of (8) occurs when the condition (9) is satisfied.

Lemma 3: Assume that it is supposed to distribute n vertices between ω complete components. The i^{th} component has c_i vertices so that

$$0 < c_1 \le c_2 \le \ldots \le c_\omega, \quad n \ge \omega \tag{14}$$

Where c_i is the number of allocated vertices to i^{th} component that $1 \le i \le \omega$.

Therefore the minimum value of (15)

$$\binom{n}{2} - \left(\sum_{i=1}^{\omega}\binom{c_i}{2}\right) + \binom{c_\omega}{2} \tag{15}$$

Occurs when vertices are placed based on (16).

$$\begin{cases} c_{\omega-1} = c_\omega = \dfrac{n-\omega+2}{2}, & c_i = 1, 1 \le i \le \omega-2 \quad if \quad \operatorname{mod}(n+\omega,2)=0 \\ c_{\omega-1} = \dfrac{n-\omega+1}{2}, c_\omega = \dfrac{n-\omega+3}{2}, & c_i = 1, 1 \le i \le \omega-2 \quad if \quad \operatorname{mod}(n+\omega,2)=1 \end{cases} \tag{16}$$

Proof: Rearranging (15) yields

$$\binom{n}{2} - \left(\sum_{i=1}^{\omega}\binom{c_i}{2}\right) + \binom{c_\omega}{2} = \binom{n}{2} - \sum_{i=1}^{\omega-1}\binom{c_i}{2}, \tag{17}$$

Where n is constant. To minimize (17), the value of $\sum_{i=1}^{\omega-1}\binom{c_i}{2}$ should be maximum. According to Lemma 3, this occurs when vertices are placed based on the following arrangement.

$$c_i = 1, \quad 1 \le i \le \omega-2, \quad c_{\omega-1} = (n-c_\omega) - (\omega-1) + 1\,. \tag{18}$$

To maximize the value of $\sum_{i=1}^{\omega-1}\binom{c_i}{2}$, $c_{\omega-1}$ should be maximum. It is assumed that n and ω are constants. So, c_ω, which must satisfy (14), should be minimal. Therefore, each of the $c_i = 1$, $1 \le i \le \omega-2$, must take one vertex and the remaining vertices distribute almost equally between $c_{\omega-1}$ and c_ω. If $n+\omega$ is even, then the number of remaining vertices namely $n-\omega+2$ is even too. Because $(n+\omega) + (n-\omega+2) = 2n+2$ is even. In this case, $c_{\omega-1}$ and c_ω should take the same value. If $n+\omega$ is odd, then the value of c_ω should be one more than $c_{\omega-1}$.

Now we return to the proof of Theorem 2.

Proof: First, it is assumed that ω be the number of generated components. To minimize the edge-tenacity with respect to ω, it is enough to minimize the numerator of the fraction of (3).

Based on Lemma 1, the numerator of (3) can be rearranged as follows:

$$f(\omega)=\binom{n}{2}-\left(\sum_{i=1}^{\omega}\binom{c_i}{2}\right)+\binom{c_\omega}{2} \tag{19}$$

Based on lemma 3, we have $c_i=1,\ 1\le i\le \omega-2$. Therefore, (19) can be written in a simpler form as follows:

$$f(\omega)=\binom{n}{2}-\binom{c_{\omega-1}}{2}. \tag{20}$$

If n+ω is even, then based on (16) edge tenacity will be calculated as follows:

$$T_e(K_n,\omega)=\frac{\binom{n}{2}-\binom{\frac{n-\omega+2}{2}}{2}}{\omega}=\frac{n(n-1)-\frac{1}{4}(n-\omega+2)(n-\omega)}{2\omega} \tag{21}$$
$$=\frac{4n^2-4n-n^2+2n\omega-\omega^2-2n+2\omega}{8\omega}=\frac{3n^2-6n}{8\omega}-\frac{\omega}{8}+\frac{n+1}{4}.$$

We are looking for the proper value of ω to minimize $T_e(K_n,\omega)$. We note in (21) that the value of $3n^2-6n$ is positive for all value of n where n>1, so by increasing the value of ω, $T_e(K_n,\omega)$ will decrease. Therefore, $T_e(K_n,\omega)$ will be minimal where ω achieves its maximum value. It should be noted that the maximum value of ω is n. Therefore, the edge-tenacity is as follows:

$$T_e(K_n)=\min_{\omega}\{T_e(K_n,\omega)\}=\frac{\binom{n}{2}}{n}=\frac{n-1}{2}. \tag{22}$$

If n+ω is odd, based on (16) the edge-tenacity will be calculated as follows:

$$T_e(K_n,\omega)=\frac{\binom{n}{2}-\binom{\frac{n-\omega+1}{2}}{2}}{\omega}=\frac{n(n-1)-\frac{1}{4}(n-\omega+1)(n-\omega-1)}{2\omega} \tag{23}$$
$$=\frac{4n^2-4n-n^2+2n\omega-\omega^2+1}{8\omega}=\frac{3n^2-4n+1}{8\omega}-\frac{\omega}{8}+\frac{n}{4}.$$

We are looking for the proper value of ω to minimize $T_e(K_n,\omega)$. We note in (23) that the value of $3n^2-4n+1$ is positive for all value of n where n>1, so, by increasing the value of ω, $T_e(K_n,\omega)$ will decrease. Therefore, $T_e(K_n,\omega)$ will be minimal where ω achieves its maximum value. It should be noted that the maximum value of ω is n. By considering ω=n, the value of ω+n will be even and we can use the result value of (22) which is $\frac{n-1}{2}$.

3. CONCLUSION

In this paper, edge-tenacities of two classes of graphs are calculated. For calculating the edge tenacity of these classes, first the function in conventional edge-tenacity formula, which is supposed to be minimized, is calculated for constant number of components. Second a function with respect to the number of components and the size of components. It is shown that the size of components is a constant value and the above function may be considered as a one-variable function. This variable is just the number of components. Finding minimum of this one-variable function is well known. Our approach may be useful for calculation of tenacity and toughness in various classes of graphs.

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Authors

Abolfazl Javan:

He received his B.Sc. in Software Engineering from Ferdowsi University of Mashhad, Mashhad, Iran, and M.Sc. degree in Algorithms and Computation from University of Tehran, Tehran, Iran. He is currently a research assistant at University of Tehran. His research interests include Graphs Theory, Randomized Algorithms, Approximation Algorithms, and Bioinformatics.

Mina Akhavan:

She received her B.Sc. in Software Engineering from Shahid Beheshti University, Tehran, Iran, and M.Sc. degree in Algorithms and Computation from University of Tehran, Tehran, Iran. Her research interests include Graphs Theory, Cloud Computing, and Bioinformatics.

Ali Moeini:

He received his Ph.D. in Nonlinear Systems at the University of Sussex, UK, in 1997. He is an Associate Professor of Electrical & Computer Engineering at Faculty of Engineering Science, School of Engineering, the University of Tehran, and ad joint Professor at Department of Information Technology Management, Faculty of Management, the University of Tehran in Iran. His research interests include Formal Methods in Knowledge and Software Engineering and Knowledge Management, Soft computing Methods, Randomized Algorithms, Approximation Algorithms, and Bioinformatics. He has published many papers in Journals and Conferences on the above mentioned topics.

Dara Moazzami:

He is a theoretical computer scientist, Professor of Mathematics at University of Tehran. He received a PhD in Applied Mathematics from the University of Northeastern- Boston MA. U.S.A. in 1992, and a B.Sc. in Pure Mathematics in 1976, and M.Sc. in Applied Mathematics in 1978, from University of Quebec–Montreal Canada. He has been on the faculty of University of Tehran since 1993, Deputy of Undergraduate and Graduate studies, Department of Engineering Basic Science in 2006-2008, and Chairman of Department of Engineering Science, 2009-present. He is Editor-in-Chief Journal of Algorithms and Computation. His research area is in algorithms and complexity theory, specifically Vulnerability in Networks and tenacity parameter.