AN ALGORITHM FOR SOLVING LINEAR OPTIMIZATION PROBLEMS SUBJECTED TO THE INTERSECTION OF TWO FUZZY RELATIONAL INEQUALITIES DEFINED BY FRANK FAMILY OF T-NORMS

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ABSTRACT

Frank t-norms are parametric family of continuous Archimedean t-norms whose members are also strict functions. Very often, this family of t-norms is also called the family of fundamental t-norms because of the role it plays in several applications. In this paper, optimization of a linear objective function with fuzzy relational inequality constraints is investigated. The feasible region is formed as the intersection of two inequality fuzzy systems defined by frank family of t-norms is considered as fuzzy composition. First, the resolution of the feasible solutions set is studied where the two fuzzy inequality systems are defined with max-Frank composition. Second, some related basic and theoretical properties are derived. Then, a necessary and sufficient condition and three other necessary conditions are presented to conceptualize the feasibility of the problem. Subsequently, it is shown that a lower bound is always attainable for the optimal objective value. Also, it is proved that the optimal solution of the problem is always resulted from the unique maximum solution and a minimal solution of the feasible region. Finally, an algorithm is presented to solve the problem and an example is described to illustrate the algorithm. Additionally, a method is proposed to generate random feasible max-Frank fuzzy relational inequalities. By this method, we can easily generate a feasible test problem and employ our algorithm to it.

KEYWORDS

Fuzzy relation, fuzzy relational inequality, linear optimization, fuzzy compositions and t-norms.

1. INTRODUCTION

In this paper, we study the following linear problem in which the constraints are formed as the intersection of two fuzzy systems of relational inequalities defined by Frank family of t-norms:

$$\begin{aligned} \min \quad & Z = c^T x \\ & A\varphi x \le b^1 \\ & D\varphi x \ge b^2 \\ & x \in [0,1]^n \end{aligned} \qquad (1)$$

Where $I_1 = \{1,2,..,m_1\}$, $I_2 = \{m_1+1, m_1+2,.., m_1+m_2\}$ and $J = \{1,2,..,n\}$. $A = (a_{ij})_{m_1 \times n}$ and $D = (d_{ij})_{m_2 \times n}$ are fuzzy matrices such that $0 \le a_{ij} \le 1$ ($\forall i \in I_1$ and $\forall j \in J$) and $0 \le d_{ij} \le 1$

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($\forall i \in I_2$ and $\forall j \in J$). $b^1 = (b_i^1)_{m_1 \times 1}$ is an m_1–dimensional fuzzy vector in $[0,1]^{m_1}$ (i.e., $0 \le b_i^1 \le 1, \forall i \in I_1$), $b^2 = (b_i^2)_{m_2 \times 1}$ is an m_2–dimensional fuzzy vector in $[0,1]^{m_2}$ (i.e., $0 \le b_i^2 \le 1, \forall i \in I_2$), and c is a vector in $\mathbb{R}^n$. Moreover, "φ" is the max-Frank composition, that is, $\varphi(x,y) = T_F^s(x,y) = \log_s\left(1 + \frac{(s^x-1)(s^y-1)}{s-1}\right)$ in which $s > 0$ and $s \ne 1$.

By these notations, problem (1) can be also expressed as follows:

$$\begin{aligned} \min\ & Z = c^T x \\ & \max_{j\in J}\{T_F^s(a_{ij}, x_j)\} \le b_i^1 \ ,\ i \in I_1 \\ & \max_{j\in J}\{T_F^s(d_{ij}, x_j)\} \ge b_i^2 \ ,\ i \in I_2 \\ & x \in [0,1]^n \end{aligned} \tag{2}$$

Especially, by setting $A = D$ and $b^1 = b^2$, the above problem is converted to max-Frank fuzzy relational equations. The above definition can be extended for $s = 0$, $s = 1$ and $s = \infty$ by taking limits. So, it is easy to verify that $T_F^0(x,y) = \min\{x,y\}$, $T_F^1(x,y) = xy$ and $T_F^\infty(x,y) = \max\{x+y-1, 0\}$, that is, Frank t-norm is converted to minimum, product and Lukasiewicz t-norm, respectively. Frank family of t-norms plays a central role in the investigation of the contraposition law for QL-implications [7].

The theory of fuzzy relational equations (FRE) was firstly proposed by Sanchez and applied in problems of the medical diagnosis [41]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [37]. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [35]. We refer the reader to [27] in which the authors provided a good overview of fuzzy relational equations.

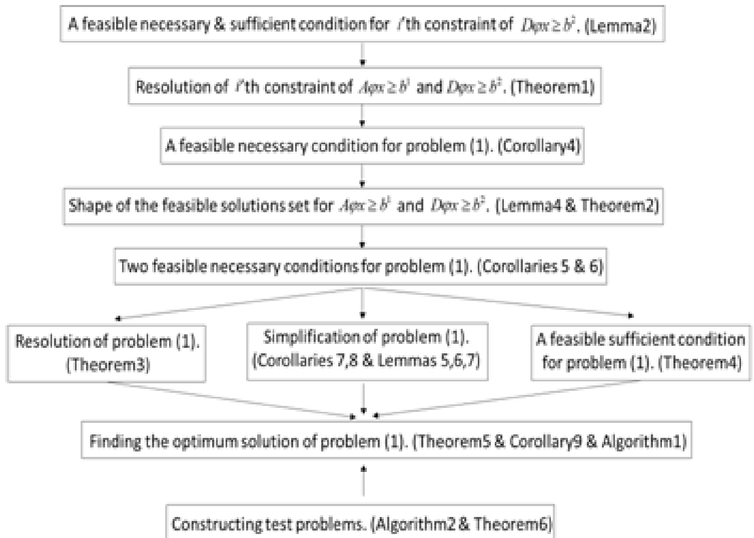
The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems. The solution set of FRE is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [5]. This non-convexity property is one of two bottlenecks making major contribution to the increase of complexity in problems that are related to FRE, especially in the optimization problems subjected to a system of fuzzy relations. The other bottleneck is concerned with detecting the minimal solutions for FREs. Chen and Wang [2] presented an algorithm for obtaining the logical representation of all minimal solutions and deduced that a polynomial-time algorithm to find all minimal solutions of FRE (with max-min composition) may not exist. In fact, the same result holds true for a more general t-norms instead of the minimum operator [2,3,30,31,34]. Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [36,38,39,42,44,45,47,50,53]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI)[13,16,17,23,28,52]. Li and Yang [28] studied a FRI with addition-min composition and presented an algorithm to search for minimal solutions. They applied FRI to meet a data transmission mechanism in a BitTorrent-like Peer-to-Peer file sharing systems. Ghodousian and Khorram [13] focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \le b^1$ and $D\varphi x \ge b^2$, and studied a mixed fuzzy system formed by the two preceding FRIs, where φ is an operator with (closed) convex solutions. Guo et al. [16] investigated a kind of FRI problems and the relationship between minimal solutions and FRI paths.

The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [1,8,11-24,25,29,32,40,43,48,52]. Fang and Li [9] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method using jump-tracking technique. Wu et al. [46] improved the method used by Fang and Li, by decreasing the search domain and presented a simplification process. Chang and Shieh [1] presented new theoretical results concerning the linear optimization problem constrained by fuzzy max–min relation equations. The topic of the linear optimization problem was also investigated with max-product operation [11,19,33]. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [22,48], max-star composition [14,24] and max-t-norm composition [20,29,43]. For example, Li and Fang [29] solved the linear optimization problem subjected to a system of sup-t equations by reducing it to a 0-1 integer optimization problem. In [20] a method was presented for solving linear optimization problems with the max-Archimedean t-norm fuzzy relation equation constraint.

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced [6,10,17,26,32,49]. For example, Wu et al. [49] represented an efficient method to optimize a linear fractional programming problem under FRE with max-Archimedean t-norm composition. Dempe and Ruziyeva [4] generalized the fuzzy linear optimization problem by considering fuzzy coefficients. Dubey et al. studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [6]. The linear optimization of bipolar FRE was studied by some researchers where FRE defined with max-min composition [10] and max-Lukasiewicz composition [26,32]. In [32], the authors presented an algorithm without translating the original problem into a 0-1 integer linear problem.

The optimization problem subjected to various versions of FRI could be found in the literature as well [12,13,16,17,23,51,52]. Yang [51] applied the pseudo-minimal index algorithm for solving the minimization of linear objective function subject to FRI with addition-min composition. Ghodousian and Khorram [12] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition. They used this fuzzy system to convincingly optimize the educational quality of a school (with minimum cost) to be selected by parents. The following diagram may help the readability of the paper.

The remainder of the paper is organized as follows. In section 2, some preliminary notions and definitions and three necessary conditions for the feasibility of problem (1) are presented. In section 3, the feasible region of problem (1) is determined as a union of the finite number of closed convex intervals. Two simplification operations are introduced to accelerate the resolution of the problem. Moreover, a necessary and sufficient condition based on the simplification operations is presented to realize the feasibility of the problem. Problem (1) is resolved by optimization of the linear objective function considered in section 4. In addition, the existence of an optimal solution is proved if problem (1) is not empty. The preceding results are summarized as an algorithm and, finally in section 5 an example is described to illustrate. Additionally, in section 5, a method is proposed to generate feasible test problems for problem (1).



2. BASIC PROPERTIES OF MAX-FRANK FRI

This section describes the basic definitions and structural properties concerning problem (1) that are used throughout the paper. For the sake of simplicity, let $S_{T_F^s}(A,b^1)$ and $S_{T_F^s}(D,b^2)$ denote the feasible solutions sets of inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, respectively, that is, $S_{T_F^s}(A,b^1)=\left\{x\in[0,1]^n \,:\, A\varphi x \leq b^1\right\}$ and $S_{T_F^s}(D,b^2)=\left\{x\in[0,1]^n \,:\, D\varphi x \geq b^2\right\}$. Also, let $S_{T_F^s}(A,D,b^1,b^2)$ denote the feasible solutions set of problem (1). Based on the foregoing notations, it is clear that $S_{T_F^s}(A,D,b^1,b^2)=S_{T_F^s}(A,b^1)\bigcap S_{T_F^s}(D,b^2)$.

Definition 1. For each $i\in I_1$ and each $j\in J$, we define $S_{T_F^s}(a_{ij},b_i^1)=\left\{x\in[0,1] : T_F^s(a_{ij},x)\leq b_i^1\right\}$. Similarly, for each $i\in I_2$ and each $j\in J$, $S_{T_F^s}(d_{ij},b_i^2)=\left\{x\in[0,1] : T_F^s(d_{ij},x)\geq b_i^2\right\}$.

Furthermore, the notations $J_i^1=\left\{j\in J \,:\, S_{T_F^s}(a_{ij},b_i^1)\neq\varnothing\right\}$, $\forall i\in I_1$, and $J_i^2=\left\{j\in J \,:\, S_{T_F^s}(d_{ij},b_i^2)\neq\varnothing\right\}$, $\forall i\in I_2$, are used in the text.

Remark 1. From the least-upper-bound property of $\mathbb{R}$, it is clear that $\inf_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}$ and $\sup_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}$ exist, if $S_{T_F^s}(a_{ij},b_i^1)\neq\varnothing$. Moreover, since T_F^s is a t-norm, its monotonicity property implies that $S_{T_F^s}(a_{ij},b_i^1)$ is actually a connected subset of $[0,1]$. Additionally, due to the continuity of T_F^s, we must have

$\inf\limits_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}=\min\limits_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}$ and $\sup\limits_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}=\max\limits_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}$.

Therefore, $S_{T_F^s}(a_{ij},b_i^1)=\left[\min\limits_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\},\max\limits_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}\right]$, i.e., $S_{T_F^s}(a_{ij},b_i^1)$ is a closed sub-interval of $[0,1]$. By the similar argument, if $S_{T_F^s}(d_{ij},b_i^2)\neq\varnothing$, then we have $S_{T_F^s}(d_{ij},b_i^2)=\left[\min\limits_{x\in[0,1]}\left\{S_{T_F^s}(d_{ij},b_i^2)\right\},\max\limits_{x\in[0,1]}\left\{S_{T_F^s}(d_{ij},b_i^2)\right\}\right]\subseteq[0,1]$.

From Definition 1 and Remark 1, the following two corollaries are resulted.

Corollary 1. For each $i\in I_1$ and each $j\in J$, $S_{T_F^s}(a_{ij},b_i^1)\neq\varnothing$. Also, $S_{T_F^s}(a_{ij},b_i^1)=\left[0,\max\limits_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}\right]$.

Corollary 2. If $S_{T_F^s}(d_{ij},b_i^2)\neq\varnothing$ for some $i\in I_2$ and $j\in J$, then $S_{T_F^s}(d_{ij},b_i^2)=\left[\min\limits_{x\in[0,1]}\left\{S_{T_F^s}(d_{ij},b_i^2)\right\},1\right]$.

Definition 2. For each $i\in I_1$ and each $j\in J$, we define

$$U_{ij}=\begin{cases}1 & a_{ij}<b_i^1\\ \log_s\left(1+\dfrac{(s^{b_i^1}-1)(s-1)}{s^{a_{ij}}-1}\right) & a_{ij}\geq b_i^1\end{cases}$$

Also, for each $i\in I_2$ and each $j\in J$, we set

$$L_{ij}=\begin{cases}+\infty & d_{ij}<b_i^2\\ 0 & d_{ij}=b_i^2=0\\ \log_s\left(1+\dfrac{(s^{b_i^2}-1)(s-1)}{s^{d_{ij}}-1}\right) & otherwise\end{cases}$$

Remark 3. From Definition 2, if $a_{ij}=b_i^1$, then $U_{ij}=1$. Also, we have $L_{ij}=1$, if $d_{ij}=b_i^2\neq 0$, and $L_{ij}=0$ if $d_{ij}>b_i^2=0$.

Lemma 1 below shows that U_{ij} and L_{ij} stated in Definition 2, determine the maximum and minimum solutions of sets $S_{T_F^s}(a_{ij},b_i^1)$ ($i\in I_1$) and $S_{T_F^s}(d_{ij},b_i^2)$ ($i\in I_2$), respectively.

Lemma 1. (a) $U_{ij} = \max_{x\in[0,1]}\left\{S_{T_F^s}(a_{ij},b_i^1)\right\}$, $\forall i \in I_1$ and $\forall j \in J$. (b) If $S_{T_F^s}(d_{ij},b_i^2) \neq \varnothing$ for some $i \in I_2$ and $j \in J$, then $L_{ij} = \min_{x\in[0,1]}\left\{S_{T_F^s}(d_{ij},b_i^2)\right\}$.

Proof. See [13,15]. □

Lemma 1 together with the corollaries 1 and 2 results in the following consequence.

Corollary 3. (a) For each $i \in I_1$ and $j \in J$, $S_{T_F^s}(a_{ij},b_i^1) = [0, U_{ij}]$. (b) If $S_{T_F^s}(d_{ij},b_i^2) \neq \varnothing$ for some $i \in I_2$ and $j \in J$, then $S_{T_F^s}(d_{ij},b_i^2) = [L_{ij},1]$.

Definition 3. For each $i \in I_1$, let $S_{T_F^s}(a_i,b_i^1) = \left\{x \in [0,1]^n : \max_{j=1}^{n}\left\{T_F^s(a_{ij},x_j)\right\} \leq b_i^1\right\}$. Similarly, for each $i \in I_2$, we define $S_{T_F^s}(d_i,b_i^2) = \left\{x \in [0,1]^n : \max_{j=1}^{n}\left\{T_F^s(d_{ij},x_j)\right\} \geq b_i^2\right\}$.

According to Definition 3 and the constraints stated in (2), sets $S_{T_F^s}(a_i,b_i^1)$ and $S_{T_F^s}(d_i,b_i^2)$ actually denote the feasible solutions sets of the i'th inequality $\max_{j\in J}\{T_F^s(a_{ij},x_j)\} \leq b_i^1$ ($i \in I_1$) and $\max_{j\in J}\{T_F^s(d_{ij},x_j)\} \geq b_i^2$ ($i \in I_2$) of problem (1), respectively. Based on (2) and Definitions 1 and 3, it can be easily concluded that for a fixed $i \in I_1$, $S_{T_F^s}(a_i,b_i^1) \neq \varnothing$ iff $S_{T_F^s}(a_{ij},b_i^1) \neq \varnothing$, $\forall j \in J$. On the other hand, by Corollary 1 we know that $S_{T_F^s}(a_{ij},b_i^1) \neq \varnothing$, $\forall i \in I_1$ and $\forall j \in J$. As a result, $S_{T_F^s}(a_i,b_i^1) \neq \varnothing$ for each $i \in I_1$. However, in contrast to $S_{T_F^s}(a_i,b_i^1)$, set $S_{T_F^s}(d_i,b_i^2)$ may be empty. Actually, for a fixed $i \in I_2$, $S_{T_F^s}(d_i,b_i^2)$ is nonempty if and only if $S_{T_F^s}(d_{ij},b_i^2)$ is nonempty for at least some $j \in J$. Additionally, for each $i \in I_2$ and $j \in J$ we have $S_{T_F^s}(d_{ij},b_i^2) \neq \varnothing$ if and only if $d_{ij} \geq b_i^2$. These results have been summarized in the following lemma. Part (b) of the lemma gives a necessary and sufficient condition for the feasibility of set $S_{T_F^s}(d_i,b_i^2)$ ($\forall i \in I_2$). It is to be noted that the lemma 2 (part (b)) also provides a necessary condition for problem (1).

Lemma 2. (a) $S_{T_F^s}(a_i,b_i^1) \neq \varnothing$, $\forall i \in I_1$. (b) For a fixed $i \in I_2$, $S_{T_F^s}(d_i,b_i^2) \neq \varnothing$ iff $\bigcup_{j=1}^{n} S_{T_F^s}(d_{ij},b_i^2) \neq \varnothing$. Additionally, for each $i \in I_2$ and $j \in J$, $S_{T_F^s}(d_{ij},b_i^2) \neq \varnothing$ iff $d_{ij} \geq b_i^2$.

Definition 4. For each $i \in I_2$ and $j \in J_i^2$, we define $S_{T_F^s}(d_i,b_i^2,j) = [0,1]\times\ldots\times[0,1]\times[L_{ij},1]\times[0,1]\times\ldots\times[0,1]$, where $[L_{ij},1]$ is in the j'th position.

In the following lemma, the feasible solutions set of the i'th fuzzy relational inequality is characterized.

Lemma 3. (a) $S_{T_F^s}(a_i, b_i^1) = [0, U_{i1}] \times [0, U_{i2}] \times \ldots \times [0, U_{in}]$, $\forall i \in I_1$. (b) $S_{T_F^s}(d_i, b_i^2) = \bigcup_{j \in J_i^2} S_{T_F^s}(d_i, b_i^2, j)$, $\forall i \in I_2$.

Proof. See [15]. □

Definition 5. Let $\overline{X}(i) = [U_{i1}, U_{i2}, \ldots, U_{in}]$, $\forall i \in I_1$. Also, let $\underline{X}(i,j) = [\underline{X}(i,j)_1, \underline{X}(i,j)_2, \ldots, \underline{X}(i,j)_n]$, $\forall i \in I_2$ and $\forall j \in J_i^2$, where

$$\underline{X}(i,j)_k = \begin{cases} L_{ij} & k = j \\ 0 & k \neq j \end{cases}$$

Lemma 3 together with Definitions 4 and 5, results in Theorem 1, which completely determines the feasible region for the i'th relational inequality.

Theorem 1. (a) $S_{T_F^s}(a_i, b_i^1) = [\mathbf{0}, \overline{X}(i)]$, $\forall i \in I_1$. (b) $S_{T_F^s}(d_i, b_i^2) = \bigcup_{j \in J_i^2} [\underline{X}(i,j), \mathbf{1}]$, $\forall i \in I_2$, where $\mathbf{0}$ and $\mathbf{1}$ are n–dimensional vectors with each component equal to zero and one, respectively.

Theorem 1 gives the upper and lower bounds for the feasible solutions set of the i'th relational inequality. Actually, for each $i \in I_1$, vectors $\mathbf{0}$ and $\overline{X}(i)$ are the unique minimum and the unique maximum of set $S_{T_F^s}(a_i, b_i^1)$. In addition, for each $i \in I_2$, set $S_{T_F^s}(d_i, b_i^2)$ has the unique maximum (i.e., vector $\mathbf{1}$), but the finite number of minimal solutions $\underline{X}(i,j)$ ($\forall j \in J_i^2$). Furthermore, part (b) of Theorem 1 presents another feasible necessary condition for problem (1) as stated in the following corollary.

Corollary 4. If $S_{T_F^s}(A, D, b^1, b^2) \neq \varnothing$, then $\mathbf{1} \in S_{T_F^s}(d_i, b_i^2)$, $\forall i \in I_2$ (i.e., $\mathbf{1} \in \bigcap_{i \in I_2} S_{T_F^s}(d_i, b_i^2) = S_{T_F^s}(D, b^2)$).

Proof. Let $S_{T_F^s}(A, D, b^1, b^2) \neq \varnothing$. Then, $S_{T_F^s}(D, b^2) \neq \varnothing$, and therefore, $S_{T_F^s}(d_i, b_i^2) \neq \varnothing$, $\forall i \in I_2$. Now, Theorem 1 (part (b)) implies $\mathbf{1} \in S_{T_F^s}(d_i, b_i^2)$, $\forall i \in I_2$. □

Lemma 4 describes the shape of the feasible solutions set for the fuzzy relational inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, separately.

Lemma 4. (a) $S_{T_F^s}(A, b^1) = \bigcap_{i \in I_1} [0, U_{i1}] \times \bigcap_{i \in I_1} [0, U_{i2}] \times \ldots \times \bigcap_{i \in I_1} [0, U_{in}]$.

(b) $S_{T_F^s}(D, b^2) = \bigcap_{i \in I_2} \bigcup_{j \in J_i^2} S_{T_F^s}(d_i, b_i^2, j)$.

Proof. The proof is obtained from Lemma 3 and equations $S_{T_F^s}(A,b^1)=\bigcap_{i\in I_1} S_{T_F^s}(a_i,b_i^1)$ and $S_{T_F^s}(D,b^2)=\bigcap_{i\in I_2} S_{T_F^s}(d_i,b_i^2)$. □

Definition 6. Let $e: I_2 \to J_i^2$ so that $e(i)=j\in J_i^2$, $\forall i\in I_2$, and let E_D be the set of all vectors e. For the sake of convenience, we represent each $e\in E_D$ as an m_2–dimensional vector $e=[j_1,j_2,...,j_{m_2}]$ in which $j_k=e(k)$, $k=1,2,...,m_2$.

Definition 7. Let $e=[j_1,j_2,...,j_{m_2}]\in E_D$. We define $\overline{X}=\min_{i\in I_1}\{\overline{X}(i)\}$, that is, $\overline{X}_j=\min_{i\in I_1}\{\overline{X}(i)_j\}$, $\forall j\in J$. Moreover, let $\underline{X}(e)=[\underline{X}(e)_1,\underline{X}(e)_2,...,\underline{X}(e)_n]$, where $\underline{X}(e)_j=\max_{i\in I_2}\{\underline{X}(i,e(i))_j\}=\max_{i\in I_2}\{\underline{X}(i,j_i)_j\}$, $\forall j\in J$.

Based on Theorem 1 and the above definition, we have the following theorem characterizing the feasible regions of the general inequalities $A\varphi x\le b^1$ and $D\varphi x\ge b^2$ in the most familiar way.

Theorem 2. (a) $S_{T_F^s}(A,b^1)=[\mathbf{0},\overline{X}]$, $\forall i\in I_1$. (b) $S_{T_F^s}(D,b^2)=\bigcup_{e\in E_D}[\underline{X}(e),\mathbf{1}]$.

Proof. For the proof in the general case see Remark 2.5 in [13]. □

Corollary 5. Assume that $S_{T_F^s}(A,D,b^1,b^2)\ne\varnothing$. Then, there exists some $e\in E_D$ such that $[\mathbf{0},\overline{X}]\cap[\underline{X}(e),\mathbf{1}]\ne\varnothing$.

Corollary 6. Assume that $S_{T_F^s}(A,D,b^1,b^2)\ne\varnothing$. Then, $\overline{X}\in S_{T_F^s}(D,b^2)$.

Proof. Let $S_{T_F^s}(A,D,b^1,b^2)\ne\varnothing$. By Corollary 5, $[\mathbf{0},\overline{X}]\cap[\underline{X}(e'),\mathbf{1}]\ne\varnothing$ for some $e'\in E_D$. Thus, $\overline{X}\in[\underline{X}(e'),\mathbf{1}]$ that means $\overline{X}\in\bigcup_{e\in E_D}[\underline{X}(e),\mathbf{1}]$. Therefore, from Theorem 2 (part (b)), $\overline{X}\in S_{T_F^s}(D,b^2)$. □

3. THE RESOLUTION OF FEASIBLE REGION AND SIMPLIFICATION OPERATIONS

In this section, two operations are presented to simplify the matrices A and D, and a necessary and sufficient condition is derived to determine the feasibility of the main problem. At first, we give a theorem in which the bounds of the feasible solutions set of problem (1) are attained. As is shown in the following theorem, by using these bounds, the feasible region is completely found. For the proof of the propositions of this section, see [13,15].

Theorem 3. Suppose that $S_{T_F^s}(A,D,b^1,b^2)\ne\varnothing$. Then $S_{T_F^s}(A,D,b^1,b^2)=\bigcup_{e\in E_D}[\underline{X}(e),\overline{X}]$.

In practice, there are often some components of matrices A and D, which have no effect on the solutions to problem (1). Therefore, we can simplify the problem by changing the values of these components to zeros. We refer the interesting reader to [13] where a brief review of such these processes is given. Here, we present two simplification techniques based on the Frank family of t-norms.

Definition 8. If a value changing in an element, say a_{ij}, of a given fuzzy relation matrix A has no effect on the solutions of problem (1), this value changing is said to be an equivalence operation.

Corollary 7. Suppose that $i \in I_1$ and $T_F^s(a_{ij_0}, x_{j_0}) < b_i, \forall x \in S_{T_F^s}(A, b^1)$. In this case, it is obvious that $\max_{j=1}^{n} \{T_F^s(a_{ij}, x_j)\} \le b_i^1$ is equivalent to $\max_{\substack{j=1 \\ j \neq j_0}}^{n} \{T_F^s(a_{ij}, x_j)\} \le b_i^1$, that is, "resetting a_{ij_0} to zero" has no effect on the solutions of problem (1) (since component a_{ij_0} only appears in the i'th constraint of problem (1)). Therefore, if $T_F^s(a_{ij_0}, x_{j_0}) < b_i^1, \forall x \in S_{T_F^s}(A, b^1)$, then "resetting a_{ij_0} to zero" is an equivalence operation.

Lemma 5 (simplification of matrix A). Suppose that matrix $\tilde{A} = (\tilde{a}_{ij})_{m_1 \times n}$ is resulted from matrix A as follows:

$$\tilde{a}_{ij} = \begin{cases} 0 & a_{ij} < b_i^1 \\ a_{ij} & a_{ij} \ge b_i^1 \end{cases}$$

for each $i \in I_1$ and $j \in J$. Then, $S_{T_F^s}(A, b^1) = S_{T_F^s}(\tilde{A}, b^1)$.

Lemma 5 gives a condition to reduce the matrix A. In this lemma, $\tilde{A}$ denote the simplified matrix resulted from A after applying the simplification process. Based on this notation, we define $\tilde{J}_i^1 = \{ j \in J : S_{T_F^s}(\tilde{a}_{ij}, b_i^1) \neq \varnothing \}$ $(\forall i \in I_1)$ where $\tilde{a}_{ij}$ denotes (i, j)'th component of matrix $\tilde{A}$. So, from Corollary 1 and Remark 2, it is clear that $\tilde{J}_i^1 = J_i^1 = J$. Moreover, since $S_{T_F^s}(A, D, b^1, b^2) = S_{T_F^s}(A, b^1) \cap S_{T_F^s}(D, b^2)$, from Lemma 5 we can also conclude that $S_{T_F^s}(A, D, b^1, b^2) = S_{T_F^s}(\tilde{A}, D, b^1, b^2)$. By considering a fixed vector $e \in E_D$ in Theorem 3, interval $[\underline{X}(e), \overline{X}]$ is meaningful iff $\underline{X}(e) \le \overline{X}$. Therefore, by deleting infeasible intervals $[\underline{X}(e), \overline{X}]$ in which $\underline{X}(e) \not\le \overline{X}$, the feasible solutions set of problem (1) stays unchanged. In order to remove such infeasible intervals from the feasible region, it is sufficient to neglect vectors e generating infeasible solutions $\underline{X}(e)$ (i.e., solutions $\underline{X}(e)$ such that $\underline{X}(e) \not\le \overline{X}$). These considerations lead us to introduce a new set $E'_D = \{ e \in E_D : \underline{X}(e) \le \overline{X} \}$ to strengthen Theorem 3. By this new set, Theorem 3 can be written as $S_{T_F^s}(A, D, b^1, b^2) = \bigcup_{e \in E'_D} [\underline{X}(e), \overline{X}]$, if $S_{T_F^s}(A, D, b^1, b^2) \neq \varnothing$.

Lemma 6. Let $I_j(e)=\{i\in I_2 : e(i)=j\}$ and $J(e)=\{j\in J : I_j(e)\neq\varnothing\}$, $\forall e\in E_D$. Then,

$$\underline{X}(e)_j=\begin{cases}\max\limits_{i\in I_j(e)}\{L_{ie(i)}\} & j\in J(e)\\ 0 & j\notin J(e)\end{cases}$$

Corollary 8. $e\in E'_D$ if and only if $L_{ie(i)}\le \overline{X}_{e(i)}$, $\forall i\in I_2$.

As mentioned before, to accelerate identification of the meaningful solutions $\underline{X}(e)$, we reduce our search to set E'_D instead of set E_D. As a result from Corollary 8, we can confine set J_i^2 by removing each $j\in J_i^2$ such that $L_{ij}>\overline{X}_j$ before selecting the vectors e to construct solutions $\underline{X}(e)$. However, lemma 7 below shows that this purpose can be accomplished by resetting some components of matrix D to zeros. Before formally presenting the lemma, some useful notations are introduced.

Definition 9 (simplification of matrix D). Let $\tilde{D}=(\tilde{d}_{ij})_{m_2\times n}$ denote a matrix resulted from D as follows:

$$\tilde{d}_{ij}=\begin{cases}0 & j\in J_i^2 \ and\ L_{ij}>\overline{X}_j\\ d_{ij} & otherwise\end{cases}$$

Also, similar to Definition 1, assume that $\tilde{J}_i^2=\left\{j\in J : S_{T_F^s}(\tilde{d}_{ij},b_i^2)\neq\varnothing\right\}$ ($\forall i\in I_2$) where $\tilde{d}_{ij}$ denotes (i,j) ‘th components of matrix $\tilde{D}$.

According to the above definition, it is easy to verify that $\tilde{J}_i^2\subseteq J_i^2$, $\forall i\in I_2$. Furthermore, the following lemma demonstrates that the infeasible solutions $\underline{X}(e)$ are not generated, if we only consider those vectors e generated by the components of the matrix $\tilde{D}$, or equivalently vectors e generated based on the set $\tilde{J}_i^2$ instead of J_i^2.

Lemma 7. $E_{\tilde{D}}=E'_D$, where $E_{\tilde{D}}$ is the set of all functions $e: I_2\to\tilde{J}_i^2$ so that $e(i)=j\in\tilde{J}_i^2$, $\forall i\in I_2$.

By Lemma 7, we always have $\underline{X}(e)\le\overline{X}$ for each vector e, which is selected based on the components of matrix $\tilde{D}$. Actually, matrix $\tilde{D}$ as a reduced version of matrix D, removes all the infeasible intervals from the feasible region by neglecting those vectors e generating the infeasible solutions $\underline{X}(e)$. Also, similar to Lemma 5 we have $S_{T_F^s}(A,D,b^1,b^2)=S_{T_F^s}(A,\tilde{D},b^1,b^2)$. This result and Lemma 5 can be summarized by $S_{T_F^s}(A,D,b^1,b^2)=S_{T_F^s}(\tilde{A},\tilde{D},b^1,b^2)$.

Definition 10. Let $L=(L_{tj})_{m_2\times n}$ be a matrix whose (i,j)'th component is equal to L_{tj}. We define the modified matrix $L^*=(L^*_{tj})_{m_2\times n}$ from the matrix L as follows:

$$L^*_{tj}=\begin{cases}+\infty & L_{tj}>\overline{X}_j\\ L_{tj} & otherwise\end{cases}$$

As will be shown in the following theorem, matrix L^* is useful for deriving a necessary and sufficient condition for the feasibility of problem (1) and accelerating identification of the set $S_{T_F^s}(A,D,b^1,b^2)$.

Theorem 4. $S_{T_F^s}(A,D,b^1,b^2)\neq\varnothing$ iff there exists at least some $j\in J_i^2$ such that $L^*_{tj}\neq+\infty$, $\forall i\in I_2$.

4. OPTIMIZATION OF THE LINEAR OBJECTIVE FUNCTION

According to the well-known schemes used for optimization of linear problems such as (1) [9,13,17,29], problem (1) is converted to the following two sub-problems:

$$(4):\ \min\ Z_1=\sum_{j=1}^{n}c_j^+x_j \qquad A\varphi x\le b^1 \qquad D\varphi x\ge b^2 \qquad x\in[0,1]^n$$

$$(5):\ \min\ Z_2=\sum_{j=1}^{n}c_j^-x_j \qquad A\varphi x\le b^1 \qquad D\varphi x\ge b^2 \qquad x\in[0,1]^n$$

Where $c_j^+=\max\{c_j,0\}$ and $c_j^-=\min\{c_j,0\}$ for $j=1,2,...,n$. It is easy to prove that $\overline{X}$ is the optimal solution of (5), and the optimal solution of (4) is $\underline{X}(e')$ for some $e'\in E'_D$.

Theorem 5. Suppose that $S_{T_F^s}(A,D,b^1,b^2)\neq\varnothing$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of sub-problems (5) and (4), respectively. Then c^Tx^* is the lower bound of the optimal objective function in (1), where $x^*=[x_1^*,x_2^*,...,x_n^*]$ is defined as follows:

$$x_j^*=\begin{cases}\overline{X}_j & c_j<0\\ \underline{X}(e^*)_j & c_j\ge0\end{cases} \qquad (6)$$

for $j=1,2,...,n$.

Proof. See Corollary 4.1 in [13]. □

Corollary 9. Suppose that $S_{T_F^s}(A,D,b^1,b^2)\neq\varnothing$. Then, $x^*=[x_1^*,x_2^*,...,x_n^*]$ as defined in (6), is the optimal solution of problem (1).

Proof. As in the poof of Theorem 5, $c^T x^*$ is the lower bound of the optimal objective function. According to the definition of vector x^*, we have $\underline{X}(e^*)_j \le x_j^* \le \overline{X}_j$, $\forall j \in J$, which implies $x^* \in \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}] = S_{T_F^s}(A, D, b^1, b^2)$. □

We now summarize the preceding discussion as an algorithm.

Algorithm 1 (solution of problem (1))

Given problem (1):

1. Compute U_{ij} ($\forall i \in I_1$ and $\forall j \in J$) and L_{ij} ($\forall i \in I_2$ and $\forall j \in J$) by Definition 2.

2. If $\mathbf{1} \in S_{T_F^s}(D, b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 4).

3. Compute vectors $\overline{X}(i)$ ($\forall i \in I_1$) from Definition 5, and then vector $\overline{X}$ from Definition 7.

4. If $\overline{X} \in S_{T_F^s}(D, b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 6).

5. Compute simplified matrices $\tilde{A}$ and $\tilde{D}$ from Lemma 5 and Definition 9, respectively.

6. Compute modified matrix L^* from Definition 10.

7. For each $i \in I_2$, if there exists at least some $j \in J_i^2$ such that $L_{ij}^* \ne +\infty$, then continue; otherwise, stop, the problem is infeasible (Theorem 4).

8. Find the optimal solution $\underline{X}(e^*)$ for the sub-problem (4) by considering vectors $e \in E_{\tilde{D}}$ and set $\tilde{J}_i^2$, $\forall i \in I_2$ (Lemma 7).

9. Find the optimal solution $x^* = [x_1^*, x_2^*, ..., x_n^*]$ for the problem (1) by (6) (Corollary 9).

It should be noted that there is no polynomial time algorithm for complete solution of FRIs with the expectation $N \ne NP$. Hence, the problem of solving FRIs is an NP-hard problem in terms of computational complexity [2].

5. Construction of Test Problems and Numerical Example

In this section, we present a method to generate random feasible regions formed as the intersection of two fuzzy inequalities with Frank family of t-norms. In section 5.1, we prove that the max-Frank fuzzy relational inequalities constructed by the introduced method are actually feasible. In section 5.2, the method is used to generate a random test problem for problem (1), and then the test problem is solved by Algorithm 1 presented in section 4.

5.1. Construction of test problems

There are several ways to generate a feasible FRI defined with max-Frank composition. In what follows, we present a procedure to generate random feasible max-Frank fuzzy relational inequalities:

Algorithm 2 (construction of feasible Max-Frank FRI)

1. Generate randon scalars $a_{ij} \in [0,1]$, $i = 1,2,...,m_1$ and $j = 1,2,...,n$, and $b_i^1 \in [0,1]$, $i = 1,2,...,m_1$.
2. Compute $\overline{X}$ by Definition 7.
2. Randomly select m_2 columns $\{ j_1, j_2,..., j_{m_2} \}$ from $J = \{1,2,...,n\}$.
2. For $i \in \{1,2,...,m_2\}$, assign a random number from $[0, \overline{X}_{j_i}]$ to b_i^2.
3. For $i \in \{1,2,...,m_2\}$, if $b_i^2 \neq 0$, then

   Assign a random number from interval $\left[\max\left\{ b_i^2, \log_s(1+\frac{(s^{b_i^2}-1)(s-1)}{(s^{\overline{X}_{j_i}}-1)}) \right\}, 1 \right]$ to d_{ij_i}.

   End
4. For $i \in \{1,2,...,m_2\}$

   For each $k \in \{1,2,...,m_2\} - \{ i \}$

   Assign a random number from $[0 , 1]$ to $d_{k j_i}$.

   End

   End
5. For each $i \in \{1,2,...,m_2\}$ and each $j \notin \{ j_1, j_2,..., j_{m_2} \}$

   Assign a random number from $[0,1]$ to d_{ij}.

   End

By the following theorem, it is proved that Algorithm 2 always generates random feasible max-Frank fuzzy relational inequalities.

Theorem 6. Problem (1) with feasible region constructed by Algorithm (2) has the nonempty feasible solutions set (i.e., $S_{T_F^s}(A, D, b^1, b^2) \neq \varnothing$).

Proof. By considering the columns $\{ j_1, j_2,..., j_{m_2} \}$ selected by Algorithm 2, let $e' = [j_1, j_2,..., j_{m_2}]$. We show that $e' \in E_D$ and $\underline{X}(e') \leq \overline{X}$. Then, the result follows from Corollary 5. From Algorithm 2, the following inequalities are resulted for each $i \in I_2$:

(I) $b_i^2 \leq \overline{X}_{j_i}$.

(II) $b_i^2 \leq d_{ij_i}$.

(III) $\log_s(1+\frac{(s^{b_i^2}-1)(s-1)}{(s^{\overline{X}_{j_i}}-1)}) \leq d_{ij_i}$.

By (I), we have $\log_s(1+\frac{(s^{b_i^2}-1)(s-1)}{(s^{\overline{X}_{j_i}}-1)}) \leq 1$. This inequality together with $b_i^2 \in [0,1]$, $\forall i \in I_2$, implies that the interval $\left[\max\left\{ b_i^2, \log_s(1+\frac{(s^{b_i^2}-1)(s-1)}{(s^{\overline{X}_{j_i}}-1)}) \right\}, 1 \right]$ is meaningful.

Also, by (II), $e'(i) = j_i \in J_i^2$, $\forall i \in I_2$. Therefore, $e' \in E_D$. Moreover, since the columns $\{ j_1, j_2,..., j_{m_2} \}$ are distinct, sets $I_{j_i}(e')$ ($i \in I_2$) are all singleton, i.e.,

$$I_{j_i}(e') = \{i\}, \ \forall i \in I_2 \tag{7}$$

As a result, we also have $J(e')=\{j_1,j_2,...,j_{m_2}\}$ and $I_j(e')=\varnothing$ for each $j\notin\{j_1,j_2,...,j_{m_2}\}$. On the other hand, from Definition 5, we have $\underline{X}(i,e'(i))_{e'(i)}=\underline{X}(i,j_i)_{j_i}=L_{ij_i}$ and $\underline{X}(i,e'(i))_j=0$ for each $j\notin J-\{j_i\}$. This fact together with (7) and Lemma 6 implies $\underline{X}(e')_{j_i}=L_{i\,j_i}$, $\forall i\in I_2$, and $\underline{X}(e')_j=0$ for $j\notin\{j_1,j_2,...,j_{m_2}\}$. So, in order to prove $\underline{X}(e')\le\overline{X}$, it is sufficient to show that $\underline{X}(e')_{j_i}\le\overline{X}_{j_i}$, $\forall i\in I_2$. But, from Definition 2 and Remark 3,

$$\underline{X}(e')_{j_i}=L_{i\,j_i}=\begin{cases}0 & b_i^2=0\\ \log_s\left(1+\dfrac{(s^{b_i^2}-1)(s-1)}{s^{d_{ij_i}}-1}\right) & b_i^2\neq 0\end{cases}\tag{8}$$

Now, inequality (III) implies

$$\log_s(1+\frac{(s^{b_i^2}-1)(s-1)}{(s^{d_{ij_i}}-1)})\le\overline{X}_{j_i}\tag{9}$$

Therefore, by relations (8) and (9), we have $\underline{X}(e')_{j_i}\le\overline{X}_{j_i}$, $\forall i\in I_2$. This completes the proof. □

5.2. Numerical Example

Consider the following linear optimization problem (1) in which the feasible region has been randomly generated by Algorithm 2 presented in section 5.1.

$$\min\ Z=0.7358x_1+5.2422x_2-3.0487x_3-0.7754x_4+2.7865x_5+8.3467x_6$$

$$\begin{bmatrix}0.1616&0.1790&0.9810&0.4075&0.9562&0.9790\\0.7156&0.6333&0.1270&0.8841&0.1240&0.2833\\0.5777&0.6240&0.2322&0.5481&0.4708&0.1338\\0.4333&0.3279&0.0236&0.3690&0.8569&0.6853\\0.8842&0.8030&0.6074&0.2083&0.0434&0.9095\\0.3931&0.9995&0.1108&0.4409&0.6916&0.6109\end{bmatrix}\varphi x\le\begin{bmatrix}0.9000\\0.1934\\0.7544\\0.3463\\0.4186\\0.1557\end{bmatrix}$$

$$\begin{bmatrix}0.0003&0.6020&0.0959&0.4564&0.9805&0.8202\\0.5409&0.8572&0.7475&0.7930&0.2348&0.8103\\0.2077&0.9883&0.7485&0.3846&0.9130&0.5570\\0.2193&0.9040&0.5433&0.5386&0.5286&0.2630\\0.6205&0.9295&0.3381&0.9917&0.0514&0.6806\\0.3258&0.4095&0.8450&0.7552&0.7569&0.2337\end{bmatrix}\varphi x\ge\begin{bmatrix}0.0504\\0.0365\\0.1080\\0.1290\\0.0482\\0.0507\end{bmatrix}$$

$$x\in[0,1]^n$$

where $|I_1|=|I_2|=|J|=6$ and $\varphi(x,y)=T_F^s(x,y)=\log_s\left(1+\dfrac{(s^x-1)(s^y-1)}{s-1}\right)$ in which $s=2$. Moreover, $Z_1=0.7358x_1+5.2422x_2+2.7865x_5+8.3467x_6$ is the objective function of sub-problem (4) and

$Z_2 = -3.0487\,x_3 - 0.7754\,x_4$ is that of sub-problem (5). By Definition 2, matrices $U = (U_{ij})_{6\times 6}$ and $L = (L_{ij})_{6\times 6}$ are as follows:

$$U = \begin{bmatrix} 1.0000 & 1.0000 & 0.9179 & 1.0000 & 0.9420 & 0.9198 \\ 0.2909 & 0.3338 & 1.0000 & 0.2261 & 1.0000 & 0.7322 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.8274 & 1.0000 & 1.0000 & 0.9492 & 0.4163 & 0.5323 \\ 0.4834 & 0.5381 & 0.7164 & 1.0000 & 1.0000 & 0.4680 \\ 0.4477 & 0.1558 & 1.0000 & 0.3992 & 0.2452 & 0.2823 \end{bmatrix}$$

$$L = \begin{bmatrix} \infty & 0.0958 & 0.6015 & 0.1316 & 0.0518 & 0.0655 \\ 0.0791 & 0.0448 & 0.0534 & 0.0496 & 0.1953 & 0.0482 \\ 0.5869 & 0.1097 & 0.1561 & 0.3271 & 0.1217 & 0.2203 \\ 0.6505 & 0.1471 & 0.2685 & 0.2710 & 0.2766 & 0.5536 \\ 0.0884 & 0.0532 & 0.1746 & 0.0488 & 0.9536 & 0.0791 \\ 0.1905 & 0.1492 & 0.0634 & 0.0731 & 0.0729 & 0.2671 \end{bmatrix}$$

Therefore, by Corollary 3 we have, for example:
$S_{T_F^s}(a_{11}, b_1^1) = [0, U_{11}] = [0,1]$ and $S_{T_F^s}(a_{45}, b_4^1) = [0, U_{45}] = [0, 0.4163]$.
$S_{T_F^s}(d_{23}, b_2^2) = [L_{23}, 1] = [0.0534, 1]$ and $S_{T_F^s}(d_{61}, b_6^2) = [L_{61}, 1] = [0.1905, 1]$.
Also, from Definition 1, $J_1^2 = \{2,3,...,6\}$ and $J_i^2 = \{1,2,...,6\}$, for $i = 2,...,6$. Actually, $S_{T_F^s}(d_{11}, b_1^2) = \varnothing$ and $S_{T_F^s}(d_{ij}, b_i^2) \neq \varnothing$ for other cases. Moreover, $d_{ij} \geq b_i^2$, $\forall i \in \{2,3,...,6\}$ and $\forall j \in J$. For the first row of matrix D, we have $0.0003 = d_{11} < b_1^2 = 0.0504$ and $d_{1j} \geq b_1^2$, $\forall j \in J - \{1\}$. Therefore, by Lemma 2 (part (b)), $S_{T_F^s}(d_i, b_i^2) = \bigcup_{j=1}^{n} S_{T_F^s}(d_{ij}, b_i^2) \neq \varnothing$, $\forall i \in I_2$.

By Definition 5, we have

$\overline{X}(1) = [1 \quad 1 \quad 0.9179 \quad 1 \quad 0.9420 \quad 0.9198]$, $\overline{X}(2) = [0.2909 \quad 0.3338 \quad 1 \quad 0.2261 \quad 1 \quad 0.7322]$,
$\overline{X}(3) = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$, $\overline{X}(4) = [0.8274 \quad 1 \quad 1 \quad 0.9492 \quad 0.4163 \quad 0.5323]$,
$\overline{X}(5) = [0.4834 \quad 0.5381 \quad 0.7164 \quad 1 \quad 1 \quad 0.4680]$, $\overline{X}(6) = [0.4477 \quad 0.1558 \quad 1 \quad 0.3992 \quad 0.2452 \quad 0.2823]$.

Also, for example

$\underline{X}(3,1) = [0.5869 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$, $\underline{X}(3,2) = [0 \quad 0.1097 \quad 0 \quad 0 \quad 0 \quad 0]$,
$\underline{X}(3,3) = [0 \quad 0 \quad 0.1561 \quad 0 \quad 0 \quad 0]$, $\underline{X}(3,4) = [0 \quad 0 \quad 0 \quad 0.3271 \quad 0 \quad 0]$,
$\underline{X}(3,5) = [0 \quad 0 \quad 0 \quad 0 \quad 0.1217 \quad 0]$, $\underline{X}(3,6) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.2203]$.
Therefore, by Theorem 1, $S_{T_F^s}(a_i, b_i^1) = [\mathbf{0}, \overline{X}(i)]$, $\forall i \in I_1$, and for example $S_{T_F^s}(d_3, b_3^2) = \bigcup_{j=1}^{6} [\underline{X}(3, j), \mathbf{1}]$, for the third row of matrix D (i.e., $i = 3 \in I_2$).
From Corollary 4, the necessary condition holds for the feasibility of the problem. More precisely, we have

$$D\varphi\mathbf{1}=\begin{bmatrix}0.9805\\0.8572\\0.9883\\0.9040\\0.9917\\0.8450\end{bmatrix}\geq\begin{bmatrix}0.0504\\0.0365\\0.1080\\0.1290\\0.0482\\0.0507\end{bmatrix}=b^2$$

that means $\mathbf{1}\in S_{T_F^s}(D,b^2)$.

From Definition 7,

$$\overline{X}=[0.29089 \quad 0.1558 \quad 0.71635 \quad 0.22607 \quad 0.24523 \quad 0.28233]$$

which determines the feasible region of the first inequalities, i.e., $S_{T_F^s}(A,b^1)=[\mathbf{0},\overline{X}]$ (Theorem 2, part (a)). Also,

$$D\varphi\overline{X}=\begin{bmatrix}0.2392\\0.5226\\0.5233\\0.3719\\0.2263\\0.5965\end{bmatrix}\geq\begin{bmatrix}0.0504\\0.0365\\0.1080\\0.1290\\0.0482\\0.0507\end{bmatrix}=b^2$$

Therefore, we have $\overline{X}\in S_{T_F^s}(D,b^2)$, which satisfies the necessary feasibility condition stated in Corollary 6. On the other hand, from Definition 6, we have $|E_D|=38880$. Therefore, the number of all vectors $e\in E_D$ is equal to 38880. However, each solution $\underline{X}(e)$ generated by vectors $e\in E_D$ is not necessary a feasible solution. For example, for $e'=[2,3,1,6,6,4]$, we attain from Definition 7

$$\underline{X}(e')=\max_{i\in I_2}\{\underline{X}(i,e'(i))\}=\max\{\underline{X}(1,2),\underline{X}(2,3),\underline{X}(3,1),\underline{X}(4,6),\underline{X}(5,6),\underline{X}(6,4)\}$$

where

$\underline{X}(1,2)=[0 \quad 0.0958 \quad 0 \quad 0 \quad 0 \quad 0]$, $\underline{X}(2,3)=[0 \quad 0 \quad 0.0534 \quad 0 \quad 0 \quad 0]$,
$\underline{X}(3,1)=[0.5869 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$, $\underline{X}(4,6)=[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.5536]$,
$\underline{X}(5,6)=[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.0791]$, $\underline{X}(6,4)=[0 \quad 0 \quad 0 \quad 0.0731 \quad 0 \quad 0]$.

Therefore, $\underline{X}(e')=[0.5869 \quad 0.0958 \quad 0.0534 \quad 0.0731 \quad 0 \quad 0.5536]$. It is obvious that $\underline{X}(e')\not\leq\overline{X}$ (actually, $\underline{X}(e')_1>\overline{X}_1$ and $\underline{X}(e')_6>\overline{X}_6$) which means $\underline{X}(e')\notin S_{T_F^s}(A,D,b^1,b^2)$ from Theorem 3. From the first simplification (Lemma 5), "resetting the following components a_{ij} to zeros" are equivalence operations: a_{11}, a_{12}, a_{14}, a_{23}, a_{25}, a_{3j} ($j=1,2,...,6$), a_{42}, a_{43}, a_{54}, a_{55}, a_{63}. So, matrix $\tilde{A}$ is resulted as follows:

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0.9810 & 0 & 0.9562 & 0.9790 \\ 0.7156 & 0.6333 & 0 & 0.8841 & 0 & 0.2833 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4333 & 0 & 0 & 0.3690 & 0.8569 & 0.6853 \\ 0.8842 & 0.8030 & 0.6074 & 0 & 0 & 0.9095 \\ 0.3931 & 0.9995 & 0 & 0.4409 & 0.6916 & 0.6109 \end{bmatrix}$$

Also, by Definition 9, we can change the value of components d_{31}, d_{34}, d_{41}, d_{44}, d_{45}, d_{46}, d_{55} to zeros. For example, since $5 \in J_4^2$ and $L_{45} = 0.2766 > 0.24523 = \overline{X}_5$, then $\tilde{d}_{45} = 0$. Simplified matrix $\tilde{D}$ is obtained as follows:

$$\tilde{D} = \begin{bmatrix} 0.0003 & 0.6020 & 0.0959 & 0.4564 & 0.9805 & 0.8202 \\ 0.5409 & 0.8572 & 0.7475 & 0.7930 & 0.2348 & 0.8103 \\ 0 & 0.9883 & 0.7485 & 0 & 0.9130 & 0.5570 \\ 0 & 0.9040 & 0.5433 & 0 & 0 & 0 \\ 0.6205 & 0.9295 & 0.3381 & 0.9917 & 0 & 0.6806 \\ 0.3258 & 0.4095 & 0.8450 & 0.7552 & 0.7569 & 0.2337 \end{bmatrix}$$

Additionally, $\tilde{J}_1^2 = \{2,3,...,6\}$, $\tilde{J}_2^2 = \{1,2,...,6\}$, $\tilde{J}_3^2 = \{2,3,5,6\}$, $\tilde{J}_4^2 = \{2,3\}$, $\tilde{J}_5^2 = \{1,2,3,4,6\}$ and $\tilde{J}_6^2 = \{1,2,...,6\}$. Based on these results and Lemma 7, we have $|E_{\tilde{D}}| = |E'_D| = 7200$. Therefore, the simplification processes reduced the number of the minimal candidate solutions from 38880 to 7200, by removing 31680 infeasible points $\underline{X}(e)$. Consequently, the feasible region has 7200 minimal candidate solutions, which are feasible. In other words, for each $e \in E_{\tilde{D}}$, we have $\underline{X}(e) \in S_{T_F^s}(A,D,b^1,b^2)$. However, each feasible solution $\underline{X}(e)$ ($e \in E_{\tilde{D}}$) may not be a minimal solution for the problem. For example, by selecting $e' = [5,2,4,1,3,6]$, we have $\underline{X}(e') = [0.0791 \quad 0.1471 \quad 0.1746 \quad 0.0731 \quad 0.0518 \quad 0.2203]$. Although $\underline{X}(e')$ is feasible (because of the inequality $\underline{X}(e') \le \overline{X}$) but it is not actually a minimal solution. To see this, let $e'' = [2,2,2,2,2,3]$. Then, $\underline{X}(e'') = [0 \quad 0.1471 \quad 0.0634 \quad 0 \quad 0 \quad 0]$. Obviously, $\underline{X}(e'') \le \underline{X}(e')$ which shows that $\underline{X}(e')$ is not a minimal solution.

Now, we obtain the modified matrix L^* according to Definition 10:

$$L^* = \begin{bmatrix} \infty & 0.0958 & 0.6015 & 0.1316 & 0.0518 & 0.0655 \\ 0.0791 & 0.0448 & 0.0534 & 0.0496 & 0.1953 & 0.0482 \\ \infty & 0.1097 & 0.1561 & \infty & 0.1217 & 0.2203 \\ \infty & 0.1471 & 0.2685 & \infty & \infty & \infty \\ 0.0884 & 0.0532 & 0.1746 & 0.0488 & \infty & 0.0791 \\ 0.1905 & 0.1492 & 0.0634 & 0.0731 & 0.0729 & 0.2671 \end{bmatrix}$$

As is shown in matrix L^*, for each $i \in I_2$ there exists at least some $j \in J_i^2$ such that $L_{ij}^* \neq +\infty$. Thus, by Theorem 4 we have $S_{T_F^s}(A,D,b^1,b^2) \neq \varnothing$.

Finally, vector $\overline{X}$ is optimal solution of sub-problem (5). For this solution, $Z_2 = \sum_{j=1}^{n} c_j^- \overline{X}_j = -3.0487\ \overline{X}_3 - 0.7754\ \overline{X}_4 = -2.3594$. Also, $Z = c^T \overline{X} = 1.7114$. In order to find the optimal solution $\underline{X}(e^*)$ of sub-problems (4), we firstly compute all minimal solutions by making pairwise comparisons between all solutions $\underline{X}(e)$ ($\forall e \in E_{\tilde{D}}$), and then we find $\underline{X}(e^*)$ among the resulted minimal solutions. Actually, the feasible region has 11 minimal solutions as follows:

$e_1 = [3\ ,3, 3\ ,\ 3\ ,\ 3\ ,\ 3]$
$\underline{X}(e_1) = [0 \quad 0 \quad 0.6015 \quad 0 \quad 0 \quad 0]$

$e_2 = [4\ ,3, 3\ ,\ 3\ ,\ 3\ ,\ 3]$
$\underline{X}(e_2) = [0 \quad 0 \quad 0.2685 \quad 0.1316 \quad 0 \quad 0]$

$e_3 = [5\ ,3, 3\ ,\ 3\ ,\ 3\ ,\ 3]$
$\underline{X}(e_3) = [0 \quad 0 \quad 0.2685 \quad 0 \quad 0.0518 \quad 0]$

$e_4 = [2\ ,2, 3\ ,\ 3\ ,\ 2\ ,\ 3]$
$\underline{X}(e_4) = [0 \quad 0.0958 \quad 0.2685 \quad 0 \quad 0 \quad 0]$

$e_5 = [6\ ,3, 3\ ,\ 3\ ,\ 3\ ,\ 3]$
$\underline{X}(e_5) = [0 \quad 0 \quad 0.2685 \quad 0 \quad 0 \quad 0.0655]$

$e_6 = [2\ ,2, 2\ ,\ 2\ ,\ 2\ ,\ 3]$
$\underline{X}(e_6) = [0 \quad 0.1471 \quad 0.0634 \quad 0 \quad 0 \quad 0]$

$e_7 = [2\ ,2, 2\ ,\ 2\ ,\ 2\ ,\ 4]$
$\underline{X}(e_7) = [0 \quad 0.1471 \quad 0 \quad 0.0731 \quad 0 \quad 0]$

$e_8 = [2\ ,2, 2\ ,\ 2\ ,\ 2\ ,\ 2]$
$\underline{X}(e_8) = [0 \quad 0.1492 \quad 0 \quad 0 \quad 0 \quad 0]$

$e_9 = [2\ ,1, 2\ ,\ 2\ ,\ 1\ ,\ 1]$
$\underline{X}(e_9) = [0.1905 \quad 0.1471 \quad 0 \quad 0 \quad 0 \quad 0]$

$e_{10} = [2\ ,2, 2\ ,\ 2\ ,\ 2\ ,\ 5]$
$\underline{X}(e_{10}) = [0 \quad 0.1471 \quad 0 \quad 0 \quad 0.0729 \quad 0]$

$e_{11} = [2\ ,2, 2\ ,\ 2\ ,\ 2\ ,\ 6]$
$\underline{X}(e_{11}) = [0 \quad 0.1471 \quad 0 \quad 0 \quad 0 \quad 0.2671]$

By comparison of the values of the objective function for the minimal solutions, $\underline{X}(e_1)$ is optimal in (4) (i.e., $e^* = e_1$). For this solution,

$$Z_1 = \sum_{j=1}^{n} c_j^+ \underline{X}(e_1)_j = 0.7358\underline{X}(e_1)_1 + 5.2422\underline{X}(e_1)_2 + 2.7865\ \underline{X}(e_1)_5 + 8.3467\underline{X}(e_1)_6 = 0.$$

Also, $Z = c^T \underline{X}(e_1) = -1.8337$. Thus, from Corollary 9, $x^* = [0 \quad 0 \quad 0.7164 \quad 0.2261 \quad 0 \quad 0]$ and then $Z^* = c^T x^* = -2.3592$.

6. CONCLUSIONS

In this paper, we proposed an algorithm to find the optimal solution of linear problems subjected to two fuzzy relational inequalities with Frank family of t-norms. The feasible solutions set of the problem is completely resolved and a necessary and sufficient condition and three necessary conditions were presented to determine the feasibility of the problem. Moreover, two simplification operations (depending on the max-Frank composition) were proposed to accelerate the solution of the problem. Finally, a method was introduced for generating feasible random max-Frank inequalities. This method was used to generate a test problem for our algorithm. The resulted test problem was then solved by the proposed algorithm. As future works, we aim at testing our algorithm in other type of linear optimization problems whose constraints are defined as FRI with other well-known t-norms.

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