THERMAL ENTANGLEMENT OF A QUBIT-QUTRIT CHAIN

S. JAMI[1,*], Z. AMERIAN[1]

[1.] Department of Physics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.

ABSTRACT

In this paper we study the effect of temperature, magnetic field, and exchange coupling on the thermal entanglement in a spin chain which consist of two qubits and one qutrit. We use negativity as a measure of entanglement in our study. We apply magnetic field, uniform and nonuniform field, on it. The results show that the entanglement decreases with increase in temperature. Also, we have found that under a magnetic field, either uniform or nonuniform, in constant temperature, the entanglement decreases. We have found that increasing exchange coupling of any two particles decreases the entanglement of the other two particles. Finally, we have compared our system with a two-particle system and found that in presence of a magnetic field the increase in number of particles leads to the decrease in the entanglement.

KEYWORDS

Thermal entanglement; qubit; qutrit; spin chain; Negativity; Magnetic field.

1. INTRODUCTION

Entanglement is the behavior of some quantum systems which first interacts each other and then separate. In quantum mechanics knowing about one part leads to have information about the other one. At first entanglement was just a theoretical subject in quantum mechanics but recently has been found many applications in quantum information and computation such as teleportation [1], super dense coding[2], quantum computation [3,4] and some cryptographic protocols [5,6] and so on.

Since the entanglement is used as a source in the many applications, it is necessary to quantify the entanglement of a system. There are many ways to quantify the entanglement of a system. In this paper, we consider a specific measure which is called the negativity $N(\rho)$ which was shown to be an easily computable measure for pure and mixed states [7] and is based on the trace norm of the particle ρ^{T_A} for the bipartite mixed states ρ . Negativity as an entanglement measure is motivated by the Peres- Horodecki positive partial transpose separability criterion [7] and is computed as follows:

$$N(\rho) = \frac{\left\| \rho^{T_A} \right\|_1 - 1}{2} \tag{1}$$

Where ρ^{T_A} is partial transpose of ρ with respect to A party. If N>0, then the two spin state is entangled .With this definition $N(\rho)$ is the sum of the absolute values of the eigen values of ρ^{T_A} are given by

$$N(\rho) = \sum_i |\mu_i| \tag{2}$$

where μ_i is the negative eigenvalues of the ρ^{T_A} .

2. THE MODEL

We consider the following Hamiltonian for the particles as the following

$$H = \sum_{i=1}^{n} [J_1 (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + J_2 (S_i^x S_{i+2}^x + S_i^y S_{i+2}^y) + B_i S_i^z] \tag{3}$$

where $S_i^\alpha \, (\alpha = x, y, z)$ are the spin operators. Also, J_1 and J_2 are the strengths of Heisenberg interaction for nearest and next nearest neighbor. We have neglected exchange interaction term along the z-axis, the magnetic field is assumed to be along the z-axis.

In this work we put n=3, as we have three particles so that $S_i = 1/2 \quad i = (1,3)$ and $S_i = 1 \quad (i = 2)$. We consider the interaction between qubit-qutrit, such as particle No.1 with particle No.2 and particle No.2 with particle No.3 also qubit-qubit interaction, such as particle No.1 with particle No.3.

$$S_{1,3}{}^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_{1,3}{}^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_{1,3}{}^x = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_2{}^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2{}^y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2{}^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4}$$

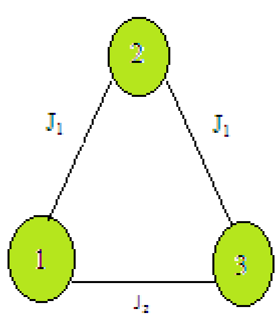


Figure 1. Diagram of the system.

Then we can write the Hamiltonian explicitly as the following

$$
\begin{aligned}
H = & J_1(S_1^x \otimes S_2^x \otimes I_3 + S_1^y \otimes S_2^y \otimes I_3 + I_1 \otimes S_2^x \otimes S_3^x + I_1 \otimes S_2^y \otimes S_3^y) \\
& + J_2(S_1^x \otimes I_2 \otimes S_3^x + S_1^y \otimes I_2 \otimes S_3^y) \\
& + (B_1 S_1^z \otimes I_2 \otimes I_3 + B_2 I_1 \otimes S_2^z \otimes I_3 + B_3 I_1 \otimes I_2 \otimes S_3^z)
\end{aligned}
\tag{5}
$$

In the above expression the sign $\otimes$ is the tensor product and I_i 's are the proper identity matrices. The calculation matrices can show that the Hamiltonian is a 12×12 matrix. On the other hand, the state of system in thermal equilibrium is

$$\rho(T) = \frac{1}{Z}\exp\left(\frac{-H}{K_B T}\right) \tag{6}$$

where $Z = Tr\left[\exp\left(-H / K_B T\right)\right]$ is the partition function, T is temperature and K_B is the Boltzmann constant and we consider $K_B = 1$. The entanglement in $\rho(T)$ is called the thermal entanglement [8].

The density matrix can be written in terms of eigenvalues and eigenvectors of Hamiltonian as the following

$$\rho(T) = \frac{1}{Z}\sum_i \exp\left(\frac{-E_i}{K_B T}\right) \left|\psi_i\right\rangle \left\langle\psi_i\right| \tag{7}$$

where $Z = Tr\left[\exp(-E_i / K_B T)|\psi_i\rangle\langle\psi_i|\right]$, E_i's are the eigenvalues and $\left|\psi_i\right\rangle$'s are the related eigenvectors of the Hamiltonian and *i* runs over the number of particles. The result of these calculations is the whole density matrix of the system. To find the entanglement between two specific particles one should trace out the other entire particles except those two particles, then use one of the measure of entanglement.

Entanglement of quantum systems changes under environmental conditions such as temperature, magnetic field, noise and so on. We have investigated the dependence of thermal entanglement in our system on temperature, external magnetic field and the effects of interaction between qubit-qutrit and qubit-qubit entanglement.

As previously mentioned, we use negativity to quantify the entanglement of system and we study behavior of negativity under changes of the above parameters. For this purpose, at first we will derive the Hamiltonian matrix from Eq. (5) and then we will obtain its eigenvalues and related eigenvectors. At the next step, by using of these eigenvalues and eigenvectors we will obtain the density matrix of the whole system ρ_{123} from Eq. (7). Then we can trace out one of the particle and find density matrix of the two other particles, such as $\rho_{12=Tr3(\rho_{123})}$.

Now we will find the eigenvalues for these density matrices ρ_{12} andρ_{13}. Finally with these eigenvalues, we will find negativity between qubit-qutrit, N_{12}, and qubit-qubit, N_{13}. It is obvious that these N_{12} and N_{13} are functions of the parameters of system, such as temperature, magnetic field and coupling constant. In this paper we study the effect of these parameters on thermal entanglement with drawing the diagram of N_{12} and N_{13} in terms of them. In the next section we do this job for both N_{12} and N_{13} .We choose magnetic field from the range of B= [-5, 5], temperature from T= (0, 0.3] and put the exchange coupling J_1 and J_2 equal to 0.5, 1 and 1.5.

3. NEGATIVITY BETWEEN QUBIT-QUTRIT (N_{12})

To calculate the negativity we consider two different cases:

3.1. Uniform magnetic field

$$\vec{B}_1 = \vec{B}_2 = \vec{B}_3 = B\,\hat{z}$$

The eigen values of Hamiltonian are as follows:

$$E_1^{\pm} = \frac{1}{4}J_2 - B \pm \frac{1}{4}\sqrt{\alpha} \quad , \quad E_2^{\pm} = \frac{1}{4}J_2 + B \pm \frac{1}{4}\sqrt{\alpha}$$

$$E_3^{\pm} = -\frac{1}{2}J_2 \pm B \quad , \quad E_4^{\pm} = \pm 2B$$

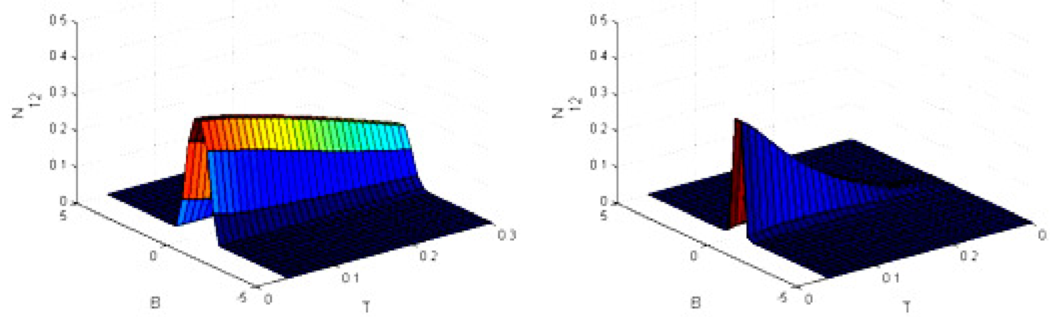
$$E_5^{\pm} = \frac{1}{4}J_2 \pm \frac{1}{4}\sqrt{\beta} \quad , \quad E_6 = -\frac{1}{2}J_2 \; , \; E_7 = 0$$

Where

$$\alpha = J_2^2 + 16J_1^2 \qquad , \quad \beta = J_2^2 + 32J_1^2$$

In Fig.[2] we plotted N_{12} negativity in terms of temperature and magnetic field for

(i) J_1=1 and J_2=0.5
(ii) J_1=1 and J_2=1.5



(a) (b)

Figure 2. N_{12} in terms B and T for J_1=1 and a) J_2=0.5 b) J_2=1.5.

It can be seen that with increasing temperature and magnetic field, entanglement decreases, furthermore with increasing J_2, the amount of entanglement decreases and the region which has nonzero entanglement becomes smaller. The center of the middle peak locates at $B = 0$.

3.2. Non-uniform magnetic field

$$\vec{B}_1 = -\vec{B}_2 = \vec{B}_3 = B\,\hat{z}$$

In this case, eigenvalues of Hamiltonian are given by

$$E_1 = E_2 = \frac{1}{4}J_2 + \frac{1}{4}\sqrt{a_1}$$

$$E_3 = E_4 = \frac{1}{4}J_2 - \frac{1}{4}\sqrt{a_2}$$

$$E_5 = \frac{1}{6}\sqrt[3]{a_4} - 6\left(-4/3B^2 - 2/3J_1^2 - 1/36J_2^3\right)/\left(1/6J_2 + \sqrt[3]{a_4}\right)$$

$$E_6^{\pm} = \frac{1}{6}\sqrt[3]{a_4} - 6\left(-4/3B^2 - 2/3J_1^2 - 1/36J_2^3\right)/\left(1/6J_2 + \sqrt[3]{a_4} \pm 1/2\sqrt{3}i\left(\frac{1}{6}\sqrt[3]{a_4} + 6\left(-4/3B^2 - 2/3J_1^2 - 1/36J_2^2\right)\right)/\sqrt[3]{a_4}\right)$$

$$E_7^{\pm} = -\frac{1}{2}J_2 \pm B \qquad , E_8 = -\frac{1}{2}J_2, \qquad E_9 = E_{10} = 0$$

Where

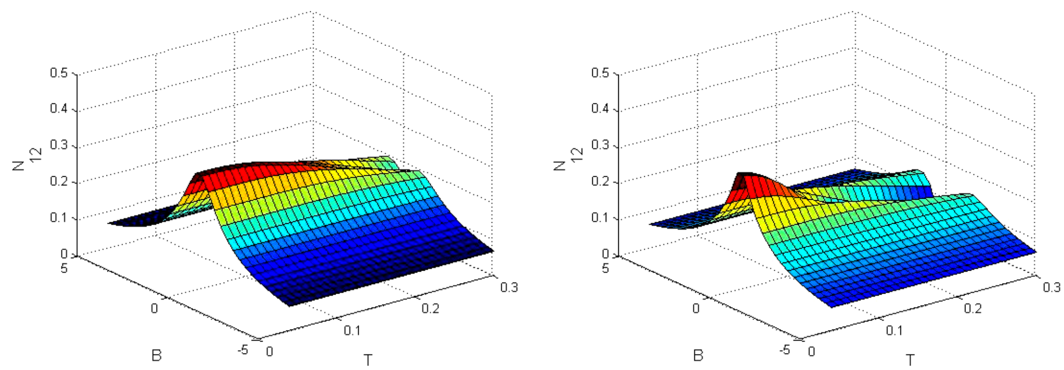
$$a_1 = J_2^2 + 16J_1^2 - 8BJ_2 + 16B^2 \qquad , \quad a_2 = J_2^2 + 16J_1^2 + 8BJ_2 + 16B^2$$

$$a_3 = -768B^6 - 1152JB^4 + 96B^4J_2^3 - 576J_1^4B^2 - 120J_1^2J_2^3B^2 - 3J_2^4B^2 - 96J_1^6 - 3J_2^3J_1^4$$

$$a_4 = -144 J_2 B^2 + 36 J_2 J_1^2 + J_2^3 + 12\sqrt{a_3}$$

In Fig.[3] we plotted N_{12} negativity in terms of temperature and magnetic field for

(i) $J_1=1$ and $J_2=0.5$
(ii) $J_1=1$ and $J_2=1.5$



(a) (b)

Figure 3. N_{12} in terms of B and T for $J_1=1$ and a) $J_2=0.5$ b) $J_2=1.5$.

We can see that as temperature increases, the value and region of entanglement (for the same B and J) getting smaller. At higher temperature, by increasing magnetic field, at first entanglement increases and then decreases.

Furthermore by comparison Fig.[3a] and [3b], we see that increasing J_2 (at fixed B and T) decreases entanglement. Comparing Fig.[2] and Fig.[3] shows that at higher temperature and at the low magnetic fields there is a minimum value for negativity in case 2 but not in case 1. Furthermore the results of two cases are independent of sign of J.

4. NEGATIVITY BETWEEN TWO QUBITS (N_{13})

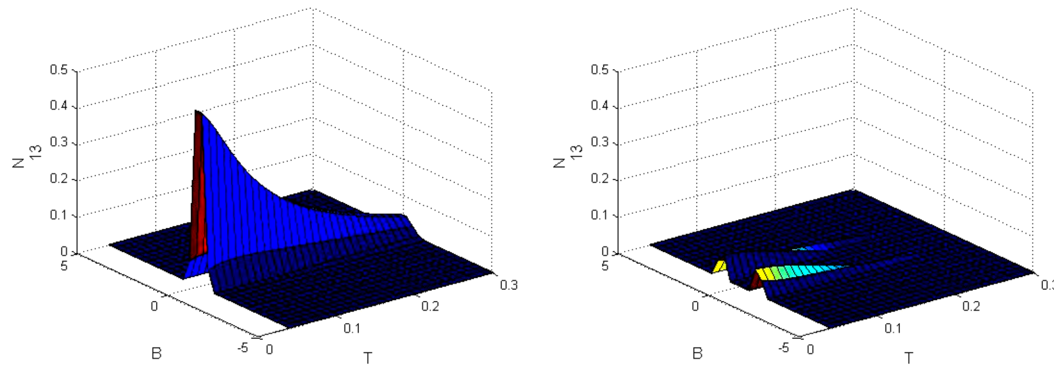
To calculate negativity, we consider two cases:

4.1. Uniform magnetic field

$$\vec{B}_1 = \vec{B}_2 = \vec{B}_3 = B\,\hat{z}$$

The result of calculations is shown in Fig.[4] for

(i) $J_2=1$, $J_1=0.5$
(ii) $J_2=1$, $J_1=1.5$.



(a) (b)

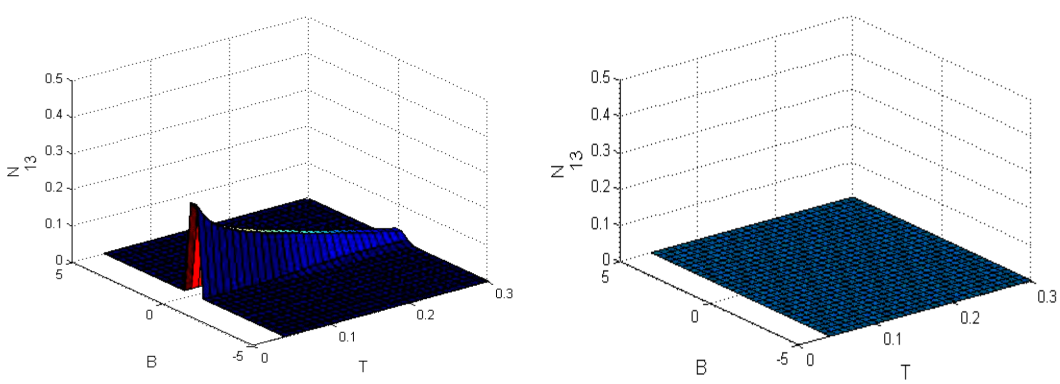
Figure 4. N_{13} in terms of B and T for J_2=1 and a) J_1=0.5 b) J_1=1.5.

Fig.[4] shows that in the case (a) negativity has a sharp peak at zero magnetic fields but in case (b) negativity vanishes at zero magnetic field and by increasing magnetic field it reaches a broad maximum.

4.2. Non-uniform magnetic field

$$\vec{B}_1 = -\vec{B}_2 = \vec{B}_3 = B\,\hat{z}$$

The results of calculation are shown in Fig.[5]. The behavior of entanglement is rather similar to the previous case (Fig.[4]). In particular for the case (b) entanglement vanishes.



(a) (b)

Figure 5. N_{13} in terms of B and T for J_2=1 and a) J_1=0.5 b) J_1=1.5.

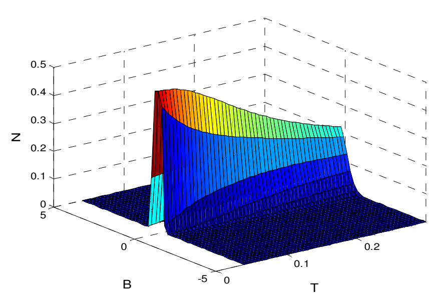
5.COMPARISON OF THREE-PARTICLE WITH TWO-PARTICLE SYSTEM

In this section, we compare a system which consists of three particles with a system consists of two particles. For this purpose we have compared N_{12} of our three–particle system with a system which consists of two particle ($S_1 = 1/2 \quad S_2 = 1$) and also we compare N_{13} of our system with a system which consists of two particles ($S_1 = 1/2 \quad S_2 = 1/2$). We do this comparison for systems under both uniform and non-uniform magnetic field.

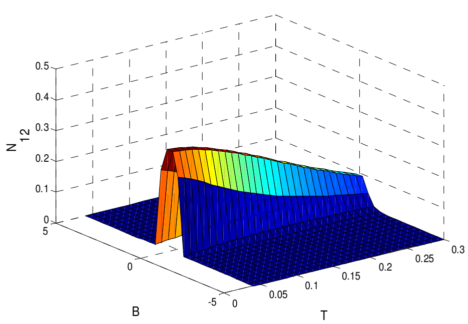
5.1.Comparing qubit-qutrit entanglement in three-particle and two-particle systems

5.1.2. Uniform magnetic field

In Fig.[6] it can be seen that, entanglement of a two –particle system is more than the related for three –particle system. Also the no vanishing region for three –particle system is smaller than the same region for two –particle system. Moreover, in high temperature, the entanglement of three – particle system goes to zero faster than two –particle system entanglement.



(a)

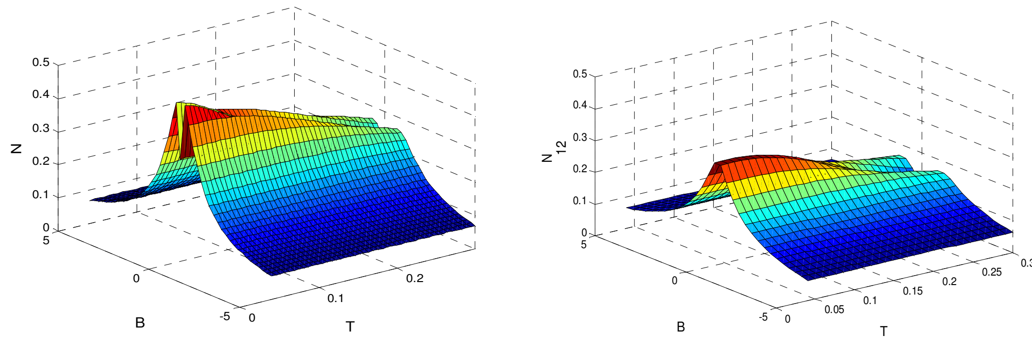


(b)

Figure 6. Entanglement in terms of B and T a) two–particle system b) three –particle system.

5.1.3. Non-uniform magnetic field

In Fig.[7], we see that, the entanglement two –particle system is more than the related for three – particle system. Fig.[7a], at low temperature and $B = 0$, we can see a minimum, which with increasing magnetic field, the entanglement increases and then decreases. But in Fig.[7b] at the same situation, we have a maximum, which with increasing magnetic field, the entanglement decreases.



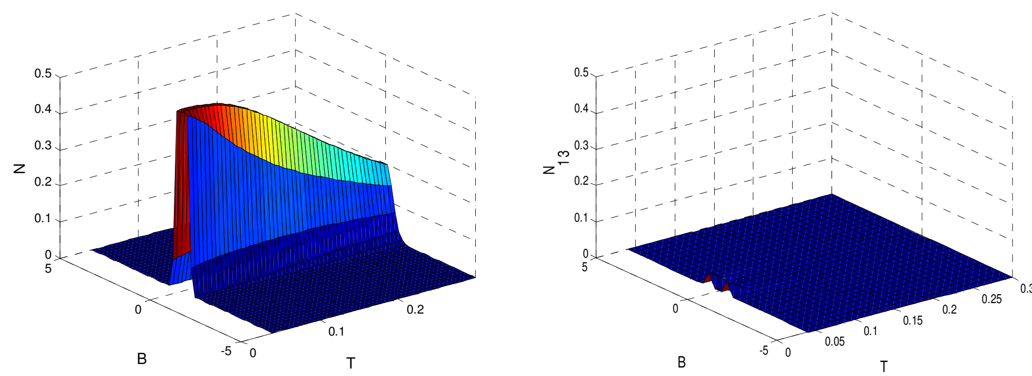
(a) (b)

Figure 7. Entanglement in terms of B and T a) two–particle system b) three –particle system.

5.2. Comparing qubit-qubit entanglement in three–particle and two-particle systems

5.2.1. Uniform magnetic field

In Fig.[8], we see that, the entanglement for two –particle system is more than the related for three –particle system. Also the no vanishing region for three –particle system is so smaller than the same region for two –particle system.

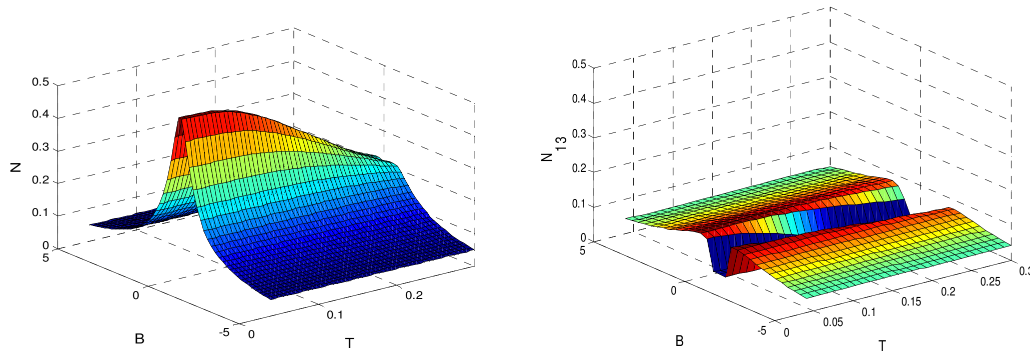


(a) (b)

Figure 8. Entanglement in terms of B and T a) two–particle system b) three –particle system.

5.2.2. Non-uniform magnetic field

In Fig.[9], we see that, the entanglement two –particle system is more than the related for three –particle system. Fig.[9a], at low temperature and $B = 0$, we can see a maximum, which with increasing magnetic field, the entanglement decreases. But in Fig.[9b] at the same situation, we have a minimum, which with increasing magnetic field, the entanglement increases and then decreases.



(a) (b)

Figure 9. Entanglement in terms of B and T a) two–particle system b) three –particle system.

6. CONCLUSION

In this paper, we have investigated the effect of temperature, magnetic field and exchange coupling on entanglement of a multiparticle system in uniform and nonuniform field. We have shown that magnetic field decreases the entanglement of our system and also with increasing temperature, the entanglement decreases. We have found that increasing exchange coupling of any two particles increases the entanglement of those two particles and decreases the entanglement of the other two particles. Also, we compare a system consists of three particles with a system consists of two particles; we found that with increasing number of particles of a system, the entanglement decreases. We found that with increasing magnetic field, the entanglement decreases, in both of our systems. But for next nearest neighbor entanglement of three particles system (qubit-qubit), we saw that with increasing magnetic field, at first enhanced and then decreased. Moreover, with increasing temperature, the entanglement of the three particles system goes to zero faster than two particles system.

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